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Modular Properties of Conditional Term Rewriting Systems

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A property of term rewriting systems is called *modular* if it is preserved under disjoint union. For unconditional term rewriting systems several modularity results are known. The aim of this paper is to analyze and extend these results to conditional term rewriting systems. It turns out that conditional term rewriting is much more complicated than unconditional rewriting from a modularity point of view. For instance, we show that the modularity of weak normalization for unconditional term rewriting systems does not extend to conditional term rewriting systems. On the positive side, we mention the extension of Toyama's confluence result for disjoint unions of term rewriting systems to conditional term rewriting systems. © 1993 Academic Press, Inc.

INTRODUCTION

Conditional term rewriting systems arise naturally in the algebraic specification of abstract data types. They have been studied by Bergstra and Klop (1986), Kaplan (1984), Kaplan and Rémy (1989), and Zhang and Rémy (1985) from this point of view. Conditional term rewriting systems are also important for integrating the functional and logic programming paradigms. Several authors have recognized that conditional term rewriting provides a natural computational mechanism for this integration; see Dershowitz and Plaisted (1985, 1987), Fribourg (1985), and Goguen and Meseguer (1986). In both uses of conditional term rewriting systems, establishing properties like confluence and strong normalization is of great importance.

Several methods are known for inferring properties of term rewriting systems like confluence and strong normalization. Generally speaking we may say that these methods have the greatest chance of succeeding if the concerned term rewriting system has few rewrite rules. For ascertaining properties of term rewriting systems with many rewrite rules it is of obvious importance to have results at our disposal which state that a term rewriting

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system has a certain property $\mathscr{P}$ if that system can be partitioned into smaller term rewriting systems which all have the property $\mathscr{P}$. For "disjoint" decompositions of term rewriting systems several positive results have been obtained. A property which is preserved under disjoint union is called *modular*. In this paper we perform a comprehensive study of conditional term rewriting systems from a modularity point of view.

The paper is organized as follows. Section 1 contains a concise introduction to conditional term rewriting. In Section 2 we pave the way for a systematic study of modularity. We give an overview of previous work on disjoint unions of term rewriting systems and we introduce the necessary technical definitions and notations for dealing with disjoint unions of conditional term rewriting systems. The research on modularity originated with Toyama (1987a) who showed that *confluence* is a modular property of term rewriting systems. In Section 3 we extend his result to join and semi-equational conditional term rewriting systems, two well-known types of conditional term rewriting systems. We also observe that *local confluence* is not a modular property of conditional term rewriting systems, notwithstanding the modularity of local confluence for unconditional term rewriting systems. In (1987b) Toyama refuted the modularity of *strong normalization*. His counterexample inspired Rusinowitch (1987) to formulate sufficient conditions for the strong normalization of the disjoint union of two strongly normalizing term rewriting systems $\mathscr{R}_1$ and $\mathscr{R}_2$ in terms of the distribution of *collapsing* and *duplicating* rules among $\mathscr{R}_1$ and $\mathscr{R}_2$. More precisely, he showed that the disjoint union of two strongly normalizing term rewriting systems $\mathscr{R}_1$ and $\mathscr{R}_2$ is strongly normalizing if neither $\mathscr{R}_1$ nor $\mathscr{R}_2$ contains collapsing rules or if both systems lack duplicating rules. Middeldorp (1989b) showed that the disjoint union of two strongly normalizing term rewriting systems is also strongly normalizing if one of the systems contains neither collapsing nor duplicating rules. For conditional term rewriting systems the situation is much more complicated as will become apparent in Section 4. We show that only one of the three sufficient conditions remains valid for conditional term rewriting systems. In order to retrieve the other two conditions we show that it is sufficient to require confluence. In Section 5 we show that the modularity of *weak normalization* for term rewriting systems does not extend to conditional term rewriting systems. We present several sufficient conditions for the modularity of weak normalization for conditional term rewriting systems. Section 6 is devoted to the modularity of *unique normal forms*. In (1989a) we proved that having unique normal forms is a modular property of term rewriting systems by showing that every term rewriting system with unique normal forms can be conservatively extended to a confluent term rewriting system with the same normal forms. We give a simple proof of this observation which facilitates the extension of the modularity of unique normal forms to semi-equational conditional term rewriting systems and we explain why this method does not work for join conditional term rewriting systems. Suggestions for further research are given in Section 7.

1. Preliminaries

Before introducing conditional term rewriting, we review the basic notions of unconditional term rewriting. Term rewriting is surveyed in Klop (1990) and Dershowitz and Jouannaud (1990).

A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $F \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countably infinite set of *variables* $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} = \varnothing$ is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F}, \mathcal{V})$ and if $F \in \mathcal{F}$ has arity n and $t_1, ..., t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $F(t_1, ..., t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. We write C instead of $C(\,)$ whenever C is a constant. The set of variables occurring in a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ is denoted by $V(t)$. Terms not containing variables are called *ground* or *closed* terms. The subset of $\mathcal{T}(\mathcal{F}, \mathcal{V})$ containing all ground terms is denoted by $\mathcal{T}(\mathcal{F})$. Identity of terms is denoted by $\equiv$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R} \subset \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$ of *rewrite rules* or *reduction rules*. Every rewrite rule (l, r) is subject to the following two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Rewrite rules (l, r) will henceforth be written as $l \to r$. We often present a TRS as a set of rewrite rules, without making explicit its signature. A rewrite rule $l \to r$ is *left-linear* if l does not contain multiple occurrences of the same variable. A *left-linear* TRS only contains left-linear rewrite rules. A rewrite rule $l \to r$ is *collapsing* if r is a variable and $l \to r$ is *duplicating* if r contains more occurrences of some variable than l.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\{x \in \mathcal{V} \mid \sigma(x) \not\equiv x\}$ is finite. This set is called the *domain* of σ and will be denoted by $\mathcal{D}(\sigma)$. Occasionally we present a substitution σ as $\{x \to \sigma(x) \mid x \in \mathcal{D}(\sigma)\}$. The *empty* substitution will be denoted by ε (here $\mathcal{D}(\varepsilon) = \varnothing$). Substitutions are extended to morphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, i.e., $\sigma(F(t_1, ..., t_n)) \equiv F(\sigma(t_1), ..., \sigma(t_n))$ for every n-ary function symbol F and terms $t_1, ..., t_n$. We call $\sigma(t)$ an *instance* of t. We frequently write t^σ instead of $\sigma(t)$. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression). If $s, t_1, ..., t_n$ are terms and $x_1, ..., x_n$ pairwise distinct variables then $s[x_i \leftarrow t_i \mid 1 \leqslant i \leqslant n]$ denotes the result of simultaneously replacing every occurrence of x_i in s by t_i $(i = 1, ..., n)$.

Let $\Box$ be a special constant symbol. A *context* $C[\ , ...,\ ]$ is a term in $\mathscr{T}(\mathscr{F} \cup \{\Box\}, \mathscr{V})$. If $C[\ , ...,\ ]$ is a context with n occurrences of $\Box$ and $t_1, ..., t_n$ are terms then $C[t_1, ..., t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1, ..., t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\ ]$. A term s is a *subterm* of a term t if there exists a context $C[\ ]$ such that $t \equiv C[s]$. We abbreviate $\mathscr{T}(\mathscr{F} \cup \{\Box\}, \mathscr{V})$ to $\mathscr{C}(\mathscr{F}, \mathscr{V})$.

The rewrite rules of a TRS $(\mathscr{F}, \mathscr{R})$ define a *rewrite relation* $\rightarrow_{\mathscr{R}}$ on $\mathscr{T}(\mathscr{F}, \mathscr{V})$ as follows: $s \rightarrow_{\mathscr{R}} t$ if there exist a rewrite rule $l \rightarrow r$ in $\mathscr{R}$, a substitution σ, and a context $C[\ ]$ such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. We say that s rewrites to t by *contracting* redex l^σ. We call $s \rightarrow_{\mathscr{R}} t$ a *rewrite step* or *reduction step*. The transitive–reflexive closure of $\rightarrow_{\mathscr{R}}$ is denoted by $\twoheadrightarrow_{\mathscr{R}}$. If $s \twoheadrightarrow_{\mathscr{R}} t$ we say that s *reduces* to t and we call t a *reduct* of s. We write $s \leftarrow_{\mathscr{R}} t$ if $t \rightarrow_{\mathscr{R}} s$; likewise for $s \twoheadleftarrow_{\mathscr{R}} t$. The transitive closure of $\rightarrow_{\mathscr{R}}$ is denoted by $\rightarrow_{\mathscr{R}}^{+}$ and $\leftrightarrow_{\mathscr{R}}$ denotes the symmetric closure of $\rightarrow_{\mathscr{R}}$ (so $\leftrightarrow_{\mathscr{R}} = \rightarrow_{\mathscr{R}} \cup \leftarrow_{\mathscr{R}}$). The transitive–reflexive closure of $\leftrightarrow_{\mathscr{R}}$ is called *conversion* and denoted by $=_{\mathscr{R}}$. If $s =_{\mathscr{R}} t$ then s and t are *convertible*. Two terms t_1, t_2 are *joinable*, notation $t_1 \downarrow_{\mathscr{R}} t_2$, if there exists a term t_3 such that $t_1 \twoheadrightarrow_{\mathscr{R}} t_3 \twoheadleftarrow_{\mathscr{R}} t_2$. Such a term t_3 is called a *common reduct* of t_1 and t_2.

A term s is a *normal form* if there is no term t with $s \rightarrow_{\mathscr{R}} t$. A term s *has* a normal form if $s \twoheadrightarrow_{\mathscr{R}} t$ for some normal form t. The set of normal forms of a TRS $(\mathscr{F}, \mathscr{R})$ is denoted by $NF(\mathscr{F}, \mathscr{R})$. When no confusion can arise, we simply write $NF(\mathscr{R})$. A TRS $(\mathscr{F}, \mathscr{R})$ is *weakly normalizing* if every term has a normal form. A TRS $(\mathscr{F}, \mathscr{R})$ is *strongly normalizing* if there are no infinite reduction sequences $t_1 \rightarrow_{\mathscr{R}} t_2 \rightarrow_{\mathscr{R}} t_3 \rightarrow_{\mathscr{R}} \cdots$. In other words, every reduction sequence eventually ends in a normal form. A TRS $(\mathscr{F}, \mathscr{R})$ is *confluent* or has the *Church–Rosser* property (CR) if for all terms s, t_1, t_2 with $t_1 \twoheadleftarrow_{\mathscr{R}} s \twoheadrightarrow_{\mathscr{R}} t_2$ we have $t_1 \downarrow_{\mathscr{R}} t_2$. A well-known equivalent formulation of confluence is that every pair of convertible terms is joinable $(t_1 =_{\mathscr{R}} t_2 \Rightarrow t_1 \downarrow_{\mathscr{R}} t_2)$. A TRS $(\mathscr{F}, \mathscr{R})$ is *locally confluent* or *weakly Church–Rosser* (WCR) if for all terms s, t_1, t_2 with $t_1 \leftarrow_{\mathscr{R}} s \rightarrow_{\mathscr{R}} t_2$ we have $t_1 \downarrow_{\mathscr{R}} t_2$. A *complete* TRS is confluent and strongly normalizing. A *semi-complete* TRS is confluent and weakly normalizing. A TRS $(\mathscr{F}, \mathscr{R})$ has *unique normal forms* (UN) if different normal forms are not convertible ($s =_{\mathscr{R}} t$ and $s, t \in NF(\mathscr{F}, \mathscr{R}) \Rightarrow s \equiv t$). The next proposition presents some of the relationships between the properties introduced so far. Part (1) is known as Newman's Lemma (1942).

PROPOSITION 1.1. (1) *Every strongly normalizing and locally confluent* TRS *is confluent.*

(2) *Every confluent* TRS *has unique normal forms.*

(3) *Every weakly normalizing* TRS *which has unique normal forms is semi-complete.* ▮

A *conditional term rewriting system* (CTRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a set $\mathcal{R}$ of *conditional rewrite rules*. Every conditional rewrite rule has the form

$$l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n$$

with $l, r, s_1, ..., s_n, t_1, ..., t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. The equations $s_1 = t_1, ..., s_n = t_n$ are the *conditions* of the rewrite rule. A rewrite rule without conditions (i.e., $n = 0$) will be written as $l \to r$. The restrictions we impose on CTRSs are the same as for unconditional TRSs: if $l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n$ is a conditional rewrite rule then l is not a single variable and variables occurring in r also occur in l. A CTRS like

$$\begin{aligned} x \leqslant x &\to true \\ x \leqslant S(x) &\to true \\ x \leqslant y &\to true \Leftarrow x \leqslant z = true, z \leqslant y = true \end{aligned}$$

with extra variables in the conditions of the rewrite rules is perfectly acceptable but due to severe technical complications we do not consider CTRSs like the following of Dershowitz, Okada, and Sivakumar (1987):

$$\begin{aligned} Fib(0) &\to \langle 0, 1 \rangle \\ Fib(S(x)) &\to \langle z, y + z \rangle \Leftarrow Fib(x) = \langle y, z \rangle. \end{aligned}$$

Depending on the interpretation of the equality sign in the conditions of the rewrite rules, different rewrite relations can be associated with a given CTRS. In this paper we restrict ourselves to the three most common interpretations.

(1) In a *join* CTRS $\mathcal{R}$ the equality sign in the conditions of the rewrite rules is interpreted as *joinability*. Formally, $s \to_{\mathcal{R}} t$ if there exist a rewrite rule $l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n$ in $\mathcal{R}$, a substitution σ, and a context $C[\ ]$ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$, and $s_i^\sigma \downarrow_{\mathcal{R}} t_i^\sigma$ for all $i \in \{1, ..., n\}$. Rewrite rules of a join CTRS will henceforth be written as

$$l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n.$$

(2) *Semi-equational* CTRSs are obtained by interpreting the equality sign in the conditions as *conversion*.

(3) In a *normal* CTRS $\mathcal{R}$ the rewrite rules are subject to the additional constraint that every t_i is a ground normal form with respect to the

unconditional TRS obtained from $\mathscr{R}$ by omitting the conditions. The rewrite relation associated with a normal CTRS is obtained by interpreting the equality sign in the conditions as *reduction* ($\twoheadrightarrow$). Rewrite rules of a normal CTRS will be presented as

$$l \to r \Leftarrow s_1 \twoheadrightarrow t_1, ..., s_n \twoheadrightarrow t_n.$$

This classification originates essentially from Bergstra and Klop (1986). The nomenclature stems from Dershowitz, Okada, and Sivakumar (1987). Due to the positiveness of the conditions in the rewrite rules of join, semi-equational, and normal CTRSs, the rewrite relation $\to_{\mathscr{R}}$ is well defined, notwithstanding the circularity in its definition. Since the rewrite relation of a normal CTRS $\mathscr{R}$ coincides with the rewrite relation of the join CTRS obtained from $\mathscr{R}$ by transforming every rewrite rule

$$l \to r \Leftarrow s_1 \twoheadrightarrow t_1, ..., s_n \twoheadrightarrow t_n$$

into

$$l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n,$$

every normal CTRS can be viewed as a join CTRS.

All notions previously defined for TRSs extend to CTRSs in the obvious way. Conditional term rewriting is inherently more complicated than ordinary term rewriting; see Bergstra and Klop (1986) and Kaplan (1984). Several well-known results for TRSs have been shown not to hold for CTRSs. Sufficient conditions for strong normalization of CTRSs were given by Kaplan (1987), Jouannaud and Waldmann (1986), and Dershowitz, Okada, and Sivakumar (1988). Sufficient conditions for confluence can be found in Bergstra and Klop (1986) and Dershowitz, Okada, and Sivakumar (1987).

EXAMPLE 1.2. The semi-equational CTRS

$$\mathscr{R}_1 = \begin{cases} a \to b \\ a \to c \\ b \to c \Leftarrow b = c \end{cases}$$

is easily shown to be confluent. So conversion in $\mathscr{R}_1$ coincides with joinability. However, the corresponding join CTRS

$$\mathscr{R}_2 = \begin{cases} a \to b \\ a \to c \\ b \to c \Leftarrow b \downarrow c \end{cases}$$

is not confluent: the reduction step from b to c is no longer allowed.

The following inductive definition of $\rightarrow_{\mathscr{R}}$ is fundamental for establishing properties of CTRSs.

DEFINITION 1.3. Let $\mathscr{R}$ be a join, semi-equational or normal CTRS. We inductively define TRSs $\mathscr{R}_i$ for $i \geqslant 0$ as follows ($\square$ denotes $\downarrow$, $=$ or $\twoheadrightarrow$):

$$\mathscr{R}_0 = \{l \rightarrow r \mid l \rightarrow r \in \mathscr{R}\}$$

$$\mathscr{R}_{i+1} = \{l^\sigma \rightarrow r^\sigma \mid l \rightarrow r \Leftarrow s_1 \square t_1, ..., s_n \square t_n \in \mathscr{R} \text{ and } s_j^\sigma \square_{\mathscr{R}_i} t_j^\sigma \text{ for } j = 1, ..., n\}.$$

Observe that $\mathscr{R}_i \subseteq \mathscr{R}_{i+1}$ for all $i \geqslant 0$. We have $s \rightarrow_{\mathscr{R}} t$ if and only if $s \rightarrow_{\mathscr{R}_i} t$ for some $i \geqslant 0$. The *depth* of a rewrite step $s \rightarrow_{\mathscr{R}} t$ is defined as the minimum i such that $s \rightarrow_{\mathscr{R}_i} t$. Depths of conversions $s =_{\mathscr{R}} t$ and "valleys" $s \downarrow_{\mathscr{R}} t$ are similarly defined.

EXAMPLE 1.4. Consider the normal CTRS

$$\mathscr{R} = \begin{cases} even(0) & \rightarrow true \\ even(S(x)) & \rightarrow odd(x) \\ odd(x) & \rightarrow true \quad \Leftarrow even(x) \twoheadrightarrow false \\ odd(x) & \rightarrow false \quad \Leftarrow even(x) \twoheadrightarrow true. \end{cases}$$

We have $even(S(0)) \rightarrow odd(0)$ by application of the second rule. The term $odd(0)$ can be further reduced to *false* by application of the last rule, using the first rule to satisfy the condition $even(0) \twoheadrightarrow true$. The depth of the rewrite step $even(0) \rightarrow true$ is 0, the depth of $even(S(0)) \rightarrow false$ is 1 and, more generally, the depth of the reduction sequence from $even(S^n(0))$ to normal form equals n for all $n \geqslant 0$.

In the sequel we make extensive use of multiset orderings.

DEFINITION 1.5. (1) A *multiset* over a set S is an unordered collection of elements of S in which elements may have multiple occurrences. To distinguish between sets and multisets we use brackets instead of braces for the latter. The set of all *finite* multisets over S is denoted by $\mathscr{M}(S)$.

(2) The *multiset extension* $\gg$ of a binary relation $>$ on a set S is a binary relation on $\mathscr{M}(S)$ defined as follows: $M_1 \gg M_2$ if there exist multisets $X, Y \in \mathscr{M}(S)$ such that

—$[\,] \neq X \subseteq M_1$,

—$M_2 = (M_1 - X) \cup Y$,

—$\forall y \in Y\ \exists x \in X$ such that $x > y$.

Occasionally we write $>^m$ instead of $\gg$.

THEOREM 1.6 (Dershowitz and Manna, 1979). *A relation $>$ on a set S is well-founded if and only if the multiset extension $\gg$ of $>$ is well-founded on $\mathcal{M}(S)$.*

2. MODULAR PROPERTIES

It is of obvious importance when by partitioning a CTRS into smaller systems the validity of a certain property for the given system can be inferred from the validity of that property for the smaller systems. This divide and conquer approach to establishing properties of CTRSs is the subject of this paper. It is very desirable when results of this kind can be obtained without imposing restrictions on the way in which systems are partitioned into smaller systems. In other words, the most useful results state that a property of CTRSs is preserved under union. Unfortunately, all interesting properties lack this behaviour. For unconditional TRSs several positive results have been obtained by imposing the disjointness requirement.

DEFINITION 2.1. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRSs with disjoint alphabets (i.e., $\mathcal{F}_1 \cap \mathcal{F}_2 = \varnothing$). The *disjoint union* $\mathcal{R}_1 \oplus \mathcal{R}_2$ of $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ is the CTRS $(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{R}_1 \cup \mathcal{R}_2)$.

DEFINITION 2.2. A property $\mathcal{P}$ of CTRSs is called *modular* if for all disjoint CTRSs $(\mathcal{F}_1, \mathcal{R}_1)$, $(\mathcal{F}_2, \mathcal{R}_2)$ the following equivalence holds:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P}$$
$$\Leftrightarrow$$
$$\text{both } (\mathcal{F}_1, \mathcal{R}_1) \text{ and } (\mathcal{F}_2, \mathcal{R}_2) \text{ have the property } \mathcal{P}.$$

In the remainder of this section we recall some of the modularity results that have been obtained for TRSs. A comprehensive survey can be found in Middeldorp (1990b). We also give the necessary technical definitions and notations for dealing with disjoint unions of CTRSs.

The research on modularity originated with Toyama (1987a) who showed the modularity of confluence. In the next section we extend this result to CTRSs.

THEOREM 2.3 (Toyama, 1987a). *Confluence is a modular property of TRSs.*

The modularity of local confluence is an easy consequence of the famous Critical Pair Lemma; see Middeldorp (1990b). In the next section we show that local confluence is not a modular property of CTRSs.

THEOREM 2.4. *Local confluence is a modular property of TRSs.*

Toyama (1987b) refuted the modularity of strong normalization by means of the following counterexample.

EXAMPLE 2.5. Let $\mathscr{R}_1 = \{F(0, 1, x) \to F(x, x, x)\}$ and

$$\mathscr{R}_2 = \begin{cases} or(x, y) \to x \\ or(x, y) \to y. \end{cases}$$

Both systems are strongly normalizing, but $\mathscr{R}_1 \oplus \mathscr{R}_2$ admits the following cyclic reduction:

$$\begin{aligned} F(or(0, 1), or(0, 1), or(0, 1)) &\to_{\mathscr{R}_1 \oplus \mathscr{R}_2} F(0, or(0, 1), or(0, 1)) \\ &\to_{\mathscr{R}_1 \oplus \mathscr{R}_2} F(0, 1, or(0, 1)) \\ &\to_{\mathscr{R}_1 \oplus \mathscr{R}_2} F(or(0, 1), or(0, 1), or(0, 1)). \end{aligned}$$

Note that $\mathscr{R}_1$ contains a duplicating rule and $\mathscr{R}_2$ consists of two collapsing rules. Observe furthermore that $\mathscr{R}_2$ is not confluent.

The next theorem states sufficient conditions for the strong normalization of $\mathscr{R}_1 \oplus \mathscr{R}_2$ in terms of the distribution of collapsing and duplicating rules among $\mathscr{R}_1$ and $\mathscr{R}_2$. The first two conditions were independently obtained by Rusinowitch (1987) and Drosten (1989). The sufficiency of the third condition is a positive answer by the present author (1989b) to a question raised in Rusinowitch (1987). In Section 4 the sufficiency of these conditions is extensively analyzed with respect to CTRSs.

THEOREM 2.6. *Suppose $\mathscr{R}_1$ and $\mathscr{R}_2$ are strongly normalizing TRSs.*

(1) *If neither $\mathscr{R}_1$ nor $\mathscr{R}_2$ contains collapsing rules then $\mathscr{R}_1 \oplus \mathscr{R}_2$ is strongly normalizing.*

(2) *If neither $\mathscr{R}_1$ nor $\mathscr{R}_2$ contains duplicating rules then $\mathscr{R}_1 \oplus \mathscr{R}_2$ is strongly normalizing.*

(3) *If one of the systems $\mathscr{R}_1, \mathscr{R}_2$ contains neither collapsing nor duplicating rules then $\mathscr{R}_1 \oplus \mathscr{R}_2$ is strongly normalizing.*

In view of Example 2.5, Toyama conjectured the modularity of completeness, but Barendregt and Klop constructed a counterexample involving a non-left-linear TRS, see Toyama (1987b). A simpler counterexample can be found in Drosten (1989). Toyama, Klop, and Barendregt (1989) gave an extremely complicated proof showing the modularity of completeness for the restricted class of left-linear TRSs. For a discussion of the next two theorems we refer to Sections 5 and 6, respectively.

THEOREM 2.7. *Weak normalization is a modular property of TRSs.*

THEOREM 2.8 (Middeldorp, 1989a). *UN is a modular property of TRSs.*

The modularity of semi-completeness is an immediate consequence of Theorems 2.3 and 2.7. We now introduce several concepts and notations for dealing with disjoint unions of CTRSs. Most of them originate from Toyama (1987a). Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be CTRSs with disjoint alphabets. Every term $t \in \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ can be viewed as an alternation of $\mathcal{F}_1$-parts and $\mathcal{F}_2$-parts. This layered structure is formalized in Definition 2.9; see Fig. 1.

Notation. We abbreviate $\mathcal{F}_1 \cup \mathcal{F}_2$ to $\mathcal{F}_\oplus$ and $\mathcal{T}(\mathcal{F}_\oplus, \mathcal{V})$ is further abbreviated to $\mathcal{T}_\oplus$. We write $\mathcal{T}_i$ instead of $\mathcal{T}(\mathcal{F}_i, \mathcal{V})$ for $i = 1, 2$. We often omit the subscript $\mathcal{R}_1 \oplus \mathcal{R}_2$ in $\rightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$, $\downarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$, and $\twoheadrightarrow_{\mathcal{R}_1 \oplus \mathcal{R}_2}$.

DEFINITION 2.9. (1) The *root symbol* of a term $t \in \mathcal{T}_\oplus$, notation *root*(t), is defined by

$$root(t) = \begin{cases} F & \text{if } \; t \equiv F(t_1, ..., t_n), \\ t & \text{if } \; t \in \mathcal{V}. \end{cases}$$

(2) Let $t \equiv C[t_1, ..., t_n]$ with $C[\,, ..., \,] \not\equiv \Box$. We write $t \equiv C[\![t_1, ..., t_n]\!]$ if $C[\,, ..., \,] \in \mathcal{C}(\mathcal{F}_a, \mathcal{V})$ and $root(t_1), ..., root(t_n) \in \mathcal{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. The t_i's are the *principal* subterms of t.

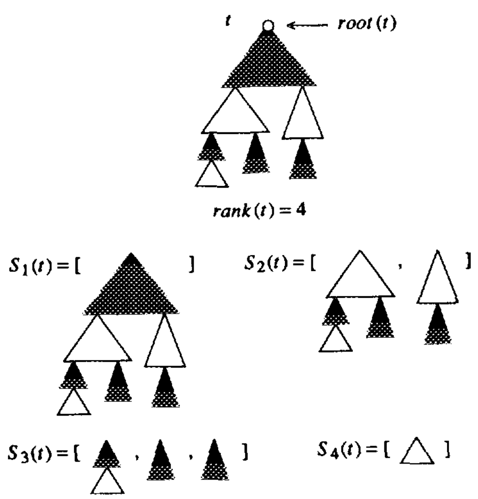

FIGURE 1

(3) The *rank* of a term $t \in \mathcal{T}_\oplus$ is defined by

$$rank(t) = \begin{cases} 1 & \text{if } \ t \in \mathcal{T}_1 \cup \mathcal{T}_2, \\ 1 + \max\{rank(t_i) \mid 1 \leqslant i \leqslant n\} & \text{if } \ t \equiv C[\![t_1, ..., t_n]\!]. \end{cases}$$

(4) The multiset $S(t)$ of *special* subterms of a term $t \in \mathcal{T}_\oplus$ is defined as follows:

$$S_1(t) = [t],$$

$$S_{n+1}(t) = \begin{cases} [\;] & \text{if } \ rank(t) = 1, \\ S_n(t_1) \cup \cdots \cup S_n(t_m) & \text{if } \ t \equiv C[\![t_1, ..., t_m]\!], \end{cases}$$

$$S(t) = \bigcup_{i \geqslant 1} S_i(t).$$

(5) The *topmost homogeneous part* of a term $t \in \mathcal{T}_\oplus$, notation $top(t)$, is the result of replacing all principal subterms of t by $\Box$; i.e.,

$$top(t) = \begin{cases} t & \text{if } \ rank(t) = 1, \\ C[\,, ..., \,] & \text{if } \ t \equiv C[\![t_1, ..., t_n]\!]. \end{cases}$$

Notation. The set $\{t \in \mathcal{T}_\oplus \mid rank(t) = n\}$ is abbreviated to $\mathcal{T}^n_\oplus$ and $\mathcal{T}^{<n}_\oplus$ denotes the set of all terms with rank less than n. We use $S_{>1}(t)$ as a shorthand for $\bigcup_{i>1} S_i(t)$.

Proposition 2.10 states some frequently used properties of special subterms. The trivial proofs are omitted.

PROPOSITION 2.10. *Let* $t \in \mathcal{T}_\oplus$.

(1) $S_n(t) = [\;] \Leftrightarrow n > rank(t)$.

(2) $S(t) = S_1(t) \cup S_{>1}(t)$.

(3) *If* $s \in S_n(t)$ *then* $rank(s) \leqslant rank(t) - n + 1$.

(4) $s \in S_2(t) \Leftrightarrow s$ *is a principal subterm of* t.

To achieve better readability we call the function symbols of $\mathcal{F}_1$ *black* and those of $\mathcal{F}_2$ *white*. Variables have no colour. A black (white) term does not contain white (black) function symbols, but may contain variables. A *top black* (*top white*) term has a black (white) root symbol. In examples, black symbols are printed as capitals and white symbols in lower case.

DEFINITION 2.11. Let $s \to t$ by application of a rewrite rule $l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n$. We write $s \to^{\mathrm{i}} t$ if the rewrite rule is being applied in one of the principal subterms of s and we write $s \to^{\mathrm{o}} t$ otherwise. The relation $\to^{\mathrm{i}}$ is called *inner* reduction and $\to^{\mathrm{o}}$ is called *outer* reduction.

Note that the inner reduction step in Fig. 2 uses a collapsing rule from $\mathscr{R}_2$ and the outer reduction step uses a duplicating rule from $\mathscr{R}_1$.

DEFINITION 2.12. We say that a rewrite step $s \to t$ is *destructive at level* 1 if the root symbols of s and t have different colours. The rewrite step $s \to t$ is *destructive at level* $n+1$ if $s \equiv C[\![s_1, ..., s_j, ..., s_n]\!] \to^{\mathrm{i}} C[s_1, ..., t_j, ..., s_n] \equiv t$ with $s_j \to t_j$ destructive at level n. Clearly, if a rewrite step is destructive then the applied rewrite rule is collapsing.

Note that $s \to t$ is destructive at level 1 if and only if $s \to^{\mathrm{o}} t$ and either $t \in V(top(s))$ or t is a principal subterm of s. It should be stressed that destructive rewrite steps at a level greater than 1 change essentially the layered structure of terms. This explains why the presence of collapsing rules is problematic from a modularity point of view.

The next definition introduces special notations for "degenerate" cases of $t \equiv C[\![t_1, ..., t_n]\!]$. Although it might give the impression of making mountains of molehills, it actually is very useful for cutting down the number of cases to be considered in many proofs in subsequent sections.

DEFINITION 2.13. First we extend the notion of context as defined in Section 1. We write $C\langle\ , ...,\ \rangle$ for a term containing zero or more occurrences of $\Box$ and $C\{\ , ...,\ \}$ denotes a term different from $\Box$ itself, containing zero or more occurrences of $\Box$. If $t \in \mathscr{T}_\oplus$ and $t_1, ..., t_n$ are the (possibly zero) principal subterms of t (from left to right), then we write $t \equiv C\{\{t_1, ..., t_n\}\}$ provided $t \equiv C\{t_1, ..., t_n\}$. We write $t \equiv C\langle\!\langle t_1, ..., t_n\rangle\!\rangle$ if $t \equiv C\langle t_1, ..., t_n\rangle$ and either $C\langle\ , ...,\ \rangle \not\equiv \Box$ and $t_1, ..., t_n$ are the principal subterms of t or $C\langle\ , ...,\ \rangle \equiv \Box$ and $t \in \{t_1, ..., t_n\}$.

The next proposition is heavily used in the sequel although this is rarely made explicit.

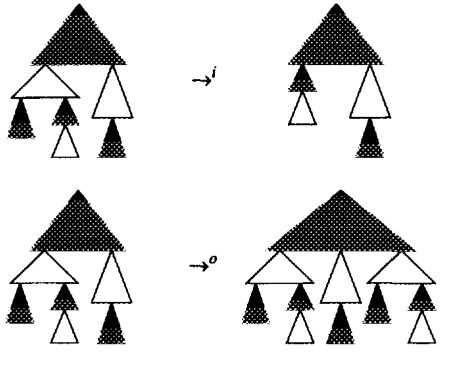

FIGURE 2

PROPOSITION 2.14. (1) *If* $s \to^{o} t$ *then* $s \equiv C\{\{s_1, ..., s_n\}\}$ *and* $t \equiv C^*\langle\langle s_{i_1}, ..., s_{i_m}\rangle\rangle$ *for some contexts* $C\{\ , ...,\ \}$ *and* $C^*\langle\ , ...,\ \rangle$, *indices* $i_1, ..., i_m \in \{1, ..., n\}$, *and terms* $s_1, ..., s_n \in \mathcal{T}_{\oplus}$. *If* $s \to^{o} t$ *is not destructive then we may write* $t \equiv C^*\{\{s_{i_1}, ..., s_{i_m}\}\}$.

(2) *If* $s \to^{i} t$ *then* $s \equiv C[\![s_1, ..., s_j, ..., s_n]\!]$ *and* $t \equiv C[s_1, ..., t_j, ..., s_n]$ *for some context* $C[\ , ...,\ ]$, *index* $j \in \{1, ..., n\}$, *and terms* $s_1, ..., s_n$, $t_j \in \mathcal{T}_{\oplus}$ *with* $s_j \to t_j$. *If* $s \to^{i} t$ *is not destructive at level* 2 *then we may write* $t \equiv C[\![s_1, ..., t_j, ..., s_n]\!]$.

Proof. Straightforward. ∎

The following proposition is very useful in proofs by induction on the rank of terms. If rewrite rules were allowed to introduce new variables, this proposition would no longer hold.

PROPOSITION 2.15. *If* $s \twoheadrightarrow t$ *then* $rank(s) \geqslant rank(t)$.

Proof. Suppose $s \to t$. Using Proposition 2.14 we obtain $rank(s) \geqslant rank(t)$ by a straightforward induction on $rank(s)$. The proposition now follows by induction on the length of $s \twoheadrightarrow t$. ∎

EXAMPLE 2.16. Consider the TRSs

$$\mathcal{R}_1 = \begin{cases} F(x, y) \to G(x) \\ G(A) \quad\ \to B \end{cases}$$

and

$$\mathcal{R}_2 = \begin{cases} e(x) \quad\ \to x \\ f(x, x) \to e(c). \end{cases}$$

In the reduction sequence

$$\begin{aligned} &F(G(e(A)), F(e(G(B)), f(e(A), e(G(c))))) \\ &\to^{i} F(G(A), F(e(G(B)), f(e(A), e(G(c))))) \\ &\to^{o} F(G(A), G(e(G(B)))) \\ &\to^{i} F(G(A), G(G(B))) \\ &\to^{o} G(G(A)) \end{aligned}$$

we have the ranks 4, 4, 3, 1, and 1 respectively. The first and third steps of this sequence are destructive at level 2.

DEFINITION 2.17. Let $s_1, ..., s_n$, $t_1, ..., t_n \in \mathcal{T}_{\oplus}$. We write $\langle s_1, ..., s_n\rangle \propto \langle t_1, ..., t_n\rangle$ if $t_i \equiv t_j$ whenever $s_i \equiv s_j$, for all $1 \leqslant i < j \leqslant n$. The combination of $\langle s_1, ..., s_n\rangle \propto \langle t_1, ..., t_n\rangle$ and $\langle t_1, ..., t_n\rangle \propto \langle s_1, ..., s_n\rangle$ is abbreviated to $\langle s_1, ..., s_n\rangle \infty \langle t_1, ..., t_n\rangle$. This notation is used to code principal subterms by variables.

PROPOSITION 2.18. *If* $C\{\{s_1, ..., s_n\}\} \to^{\circ} C^*\langle\langle s_{i_1}, ..., s_{i_m}\rangle\rangle$ *then* $C\{t_1, ..., t_n\} \to^{\circ} C^*\langle t_{i_1}, ..., t_{i_m}\rangle$ *for all terms* $t_1, ..., t_n$ *with* $\langle s_1, ..., s_n\rangle \propto \langle t_1, ..., t_n\rangle$. *Furthermore, if the applied rewrite rule is left-linear then the restriction* $\langle s_1, ..., s_n\rangle \propto \langle t_1, ..., t_n\rangle$ *can be omitted.*

Proof. Routine. ▮

DEFINITION 2.19. A term t is *root preserved* if the root symbols of t and t' have the same colour for every term t' with $t \twoheadrightarrow t'$. A term t is *preserved* if t is root preserved and every principal subterm of t is preserved. In other words, t is preserved if all special subterms of t are root preserved.

DEFINITION 2.20. Suppose σ and τ are substitutions. We write $\sigma \propto \tau$ if $x^\sigma \equiv y^\sigma$ implies $x^\tau \equiv y^\tau$ for all $x, y \in \mathscr{V}$. Note that $\sigma \propto \varepsilon$ if and only if σ is injective. We write $\sigma \twoheadrightarrow \tau$ if $x^\sigma \twoheadrightarrow x^\tau$ for all $x \in \mathscr{V}$. Clearly $t^\sigma \twoheadrightarrow t^\tau$ whenever $\sigma \twoheadrightarrow \tau$, for all $t \in \mathscr{T}_\oplus$.

DEFINITION 2.21. A substitution σ is *preserved* if x^σ is preserved for every $x \in \mathscr{D}(\sigma)$.

DEFINITION 2.22. A substitution σ is *black* (*white*) if x^σ is a black (white) term for every $x \in \mathscr{D}(\sigma)$ and σ is *top black* (*top white*) if x^σ is top black (top white) for every $x \in \mathscr{D}(\sigma)$.

PROPOSITION 2.23. *Every substitution* σ *can be decomposed into* $\sigma_2 \circ \sigma_1$ *such that* σ_1 *is black* (*white*), σ_2 *is top white* (*top black*), *and* $\sigma_2 \propto \varepsilon$.

Proof. Let $\{t_1, ..., t_n\}$ be the set of all maximal subterms of x^σ for $x \in \mathscr{D}(\sigma)$ with white root. Choose distinct fresh variables $z_1, ..., z_n$ and define the substitution σ_2 by $\sigma_2 = \{z_i \to t_i \mid 1 \leqslant i \leqslant n\}$. Let $x \in \mathscr{D}(\sigma)$. We define $\sigma_1(x)$ by case analysis (see Fig. 3).

(1) If x^σ is top white then $x^\sigma \equiv t_i$ for some $i \in \{1, ..., n\}$. In this case we define $\sigma_1(x) \equiv z_i$.

(2) If x^σ is a black term then we take $\sigma_1(x) \equiv x^\sigma$.

(3) In the remaining case we can write $x^\sigma \equiv C[\![t_{i_1}, ..., t_{i_k}]\!]$ for some $1 \leqslant i_1, ..., i_k \leqslant n$ and we define $\sigma_1(x) \equiv C[z_{i_1}, ..., z_{i_k}]$.

By construction we have $\sigma_2 \propto \varepsilon$, σ_1 is black, and σ_2 is top white. ▮

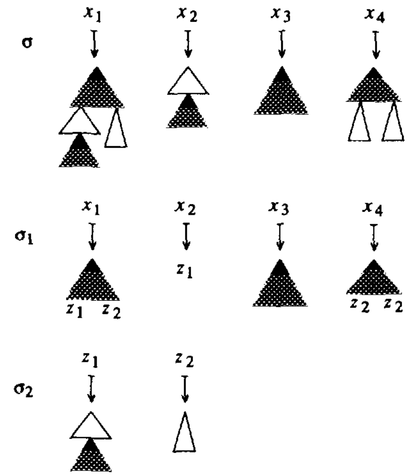

FIGURE 3

In the sequel we only state propositions for a single colour situation (usually ... *black* term ... *top white* substitution ...) without mentioning the reverse situation between parentheses.

3. CONFLUENCE[1]

In this section we first show that confluence is a modular property of join CTRSs. To this end, we assume that $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint confluent join CTRSs. The fundamental property of the disjoint union of two TRSs $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$, that is to say that $s \to_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ implies either $s \to_{\mathscr{R}_1} t$ or $s \to_{\mathscr{R}_2} t$, is not true for (join) CTRSs, as can be seen from the next example.

EXAMPLE 3.1. Let $\mathscr{R}_1 = \{F(x, y) \to G(x) \Leftarrow x \downarrow y\}$ and $\mathscr{R}_2 = \{a \to b\}$. We have $F(a, b) \to_{\mathscr{R}_1 \oplus \mathscr{R}_2} G(a)$ because $a \downarrow_{\mathscr{R}_1 \oplus \mathscr{R}_2} b$, but neither $F(a, b) \to_{\mathscr{R}_1} G(a)$ nor $F(a, b) \to_{\mathscr{R}_2} G(a)$.

The problem is that when a rule of one of the systems is being applied, rules of the other system may be needed in order to satisfy the conditions. So the question arises of how the rewrite relation $\to_{\mathscr{R}_1 \oplus \mathscr{R}_2}$ is related to $\to_{\mathscr{R}_1}$ and $\to_{\mathscr{R}_2}$. In the above example we have

$$F(a, b) \to_{\mathscr{R}_2} F(b, b) \to_{\mathscr{R}_1} G(b) \leftarrow_{\mathscr{R}_2} G(a).$$

[1] Part of the material presented in this section originates in Middeldorp (1990a).

This suggests that $\to_{\mathscr{R}_1 \oplus \mathscr{R}_2}$ corresponds to joinability with respect to the union of $\to_{\mathscr{R}_1}$ and $\to_{\mathscr{R}_2}$. However, $\to_{\mathscr{R}_1} \cup \to_{\mathscr{R}_2}$ is not an entirely satisfactory relation from a technical viewpoint. For instance, confluence of $\to_{\mathscr{R}_1} \cup \to_{\mathscr{R}_2}$ is not easily proved. We define two more manageable rewrite relations $\to_1$ and $\to_2$ such that

(1) their union is confluent (Lemma 3.6),

(2) reduction in $\mathscr{R}_1 \oplus \mathscr{R}_2$ corresponds to joinability with respect to $\to_1 \cup \to_2$ (Lemma 3.7).

From these two properties the modularity of confluence for join CTRSs is easily inferred. The proof of the first property is a more or less straightforward reduction to Theorem 2.3. The proof of the second property is rather technical but we believe that the underlying ideas are simple. Contrary to usual mathematical practice we present certain parts of our proof in a top-down fashion in order to make its structure more accessible. Figure 4 exhibits the dependencies between the various results.

DEFINITION 3.2. The rewrite relation $\to_1$ is defined as follows: $s \to_1 t$ if there exist a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, \ldots, s_n \downarrow t_n$ in $\mathscr{R}_1$, a context $C[\ ]$, and a substitution σ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$, and $s_i^\sigma \downarrow_1^o t_i^\sigma$ for $i = 1, \ldots, n$, where the superscript o in $s_i^\sigma \downarrow_1^o t_i^\sigma$ means that s_i^σ and t_i^σ are joinable using only *outer* $\to_1$-reduction steps. The relation $\to_2$ is defined similarly.

EXAMPLE 3.3. Let

$$\mathscr{R}_1 = \begin{cases} F(x, y) \to G(x) \Leftarrow x \downarrow y \\ A \to B \end{cases}$$

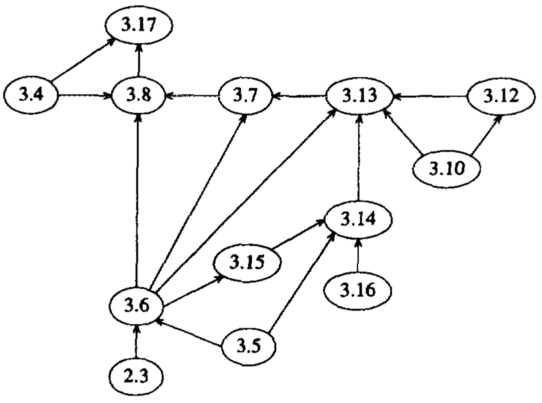

FIGURE 4

and suppose $\mathscr{R}_2$ contains a unary function symbol g. We have $F(g(A), g(B)) \to_{\mathscr{R}_1} G(g(A))$ but not $F(g(A), g(B)) \to_1 G(g(A))$ because $g(A)$ and $g(B)$ are different normal forms with respect to $\to_1^{\circ}$. The terms $F(g(A), g(B))$ and $G(g(A))$ are joinable with respect to $\to_1$:

$$F(g(A), g(B)) \to_1 F(g(B), g(B)) \to_1 G(g(B)) \leftarrow_1 G(g(A)).$$

Notation. The union of $\to_1$ and $\to_2$ is denoted by $\to_{1,2}$.

PROPOSITION 3.4. *If $s \to_{1,2} t$ then $s \to t$.*

Proof. Trivial. ∎

The next proposition states a desirable property of $\to_1^{\circ}$-reduction. The proof, however, is more complicated than the analogous statement for TRSs (cf. Proposition 2.18).

PROPOSITION 3.5. *Let s, t be black terms and suppose σ is a top white substitution such that $s^{\sigma} \to_1^{\circ} t^{\sigma}$. If τ is a substitution with $\sigma \propto \tau$ then $s^{\tau} \to_1^{\circ} t^{\tau}$.*

Proof. We prove the statement by induction on the depth of $s^{\sigma} \to_1^{\circ} t^{\sigma}$. The case of zero depth is straightforward. If the depth of $s^{\sigma} \to_1^{\circ} t^{\sigma}$ equals $n+1$ ($n \geqslant 0$) then there exist a context $C[\ ]$, a substitution ρ, and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_m \downarrow t_m$ in $\mathscr{R}_1$ such that $s^{\sigma} \equiv C[\rho(l)]$, $t^{\sigma} \equiv C[\rho(r)]$ and $\rho(s_i) \downarrow_1^{\circ} \rho(t_i)$ for $i = 1, ..., m$ with depth less than or equal to n. Proposition 2.23 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 is top white, and $\rho_2 \propto \varepsilon$. The situation is illustrated in Fig. 5. We define the substitution ρ^* by $\rho^*(x) \equiv y^{\tau}$ for every $x \in \mathscr{D}(\rho_2)$ and $y \in \mathscr{D}(\sigma)$ satisfying $\rho_2(x) \equiv y^{\sigma}$. Note that ρ^* is well-defined by the assumption $\sigma \propto \tau$. We have $\rho_2 \propto \rho^*$ since $\rho_2 \propto \varepsilon$ and $\varepsilon \propto \rho^*$. Combined with $\rho_2(\rho_1(s_i)) \downarrow_1^{\circ} \rho_2(\rho_1(t_i))$, the induction hypothesis, and the observation that if $\rho_2(u_1) \to_1^{\circ} u_2$ and u_1 is a black term then $u_2 \equiv \rho_2(u_3)$ for some black term u_3, we obtain $\rho^*(\rho_1(s_i)) \downarrow_1^{\circ} \rho^*(\rho_1(t_i))$ by a straightforward induction on the length of the conversion $\rho_2(\rho_1(s_i)) \downarrow_1^{\circ} \rho_2(\rho_1(t_i))$ for $i = 1, ..., m$; see Fig. 6. Hence

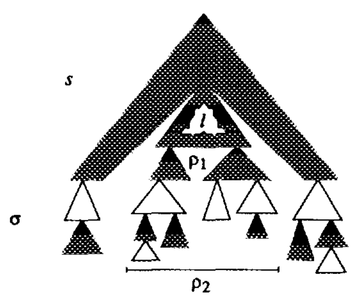

FIGURE 5

$$\begin{tikzcd}
\rho_2(\rho_1(s_i)) \arrow[r] \arrow[d, phantom, "\equiv"] & u_1 \arrow[r] \arrow[d, phantom, "\equiv"] & u_2 \arrow[d, phantom, "\equiv"] \arrow[r, phantom, "\text{observation}"] & \rho_2(\rho_1(t_i)) \arrow[l] \arrow[d, phantom, "\equiv"] \\
\rho_2(\rho_1(s_i)) \arrow[r] \arrow[d, squiggly] & \rho_2(v_1) \arrow[r] \arrow[d, squiggly] & \rho_2(v_2) \arrow[d, squiggly] \arrow[dr, phantom, "\text{hypothesis induction}"] & \rho_2(\rho_1(t_i)) \arrow[l] \arrow[d, squiggly] \\
\rho^*(\rho_1(s_i)) \arrow[r] & \rho^*(v_1) \arrow[r] & \rho^*(v_2) & \rho^*(\rho_1(t_i)) \arrow[l]
\end{tikzcd}$$

$$\longrightarrow \; \equiv \; \longrightarrow_1^{o}$$

FIGURE 6

$\rho^*(\rho_1(l)) \to_1^o \rho^*(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing every principal subterm, which has the form x^σ for some variable $x \in \mathscr{D}(\sigma)$, by the corresponding x^τ. It is not difficult to see that $s^\tau \equiv C^*[\rho^*(\rho_1(l))]$ and $t^\tau \equiv C^*[\rho^*(\rho_1(r))]$. Hence $s^\tau \to_1^o t^\tau$. ∎

LEMMA 3.6. *The relation $\to_{1,2}$ is confluent.*

Proof. We define TRSs $(\mathscr{F}_1, \mathscr{S}_1)$ and $(\mathscr{F}_2, \mathscr{S}_2)$ by $(i = 1, 2)$

$$\mathscr{S}_i = \{s \to t \mid s, t \in \mathscr{T}_i \text{ and } s \to_i t\}.$$

With some effort we can show that the restrictions of $\to_{\mathscr{S}_i}$, $\to_i$ and $\to_{\mathscr{R}_i}$ to $\mathscr{T}_i \times \mathscr{T}_i$ are the same. Therefore $(\mathscr{F}_1, \mathscr{S}_1)$ and $(\mathscr{F}_2, \mathscr{S}_2)$ are confluent TRSs. Theorem 2.3 yields the confluence of $\mathscr{S}_1 \oplus \mathscr{S}_2$. We show that the relations $\to_{\mathscr{S}_i}$ and $\to_i$ coincide on $\mathscr{T}_\oplus \times \mathscr{T}_\oplus$. Without loss of generality we only consider the case $i = 1$.

$\subseteq$ If $s \to_{\mathscr{S}_1} t$ then there exist a rewrite rule $l \to r \in \mathscr{S}_1$, a substitution σ, and a context $C[\ ]$ such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. By definition $l \to_1 r$, from which we immediately obtain $s \to_1 t$.

$\supseteq$ If $s \to_1 t$ then there exist a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n$ in $\mathscr{R}_1$, a substitution σ, and a context $C[\ ]$ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$, and $s_i^\sigma \downarrow_1^o t_i^\sigma$ for $i = 1, ..., n$. According to Proposition 2.23 we can decompose σ into $\sigma_2 \circ \sigma_1$ such that σ_1 is black, σ_2 is top white, and $\sigma_2 \propto \varepsilon$. Induction on the number of rewrite steps in $s_i^\sigma \downarrow_1^o t_i^\sigma$ together with Proposition 3.5 and the observation made in the proof of Proposition 3.5 yields $\sigma_1(s_i) \downarrow_1^o \sigma_1(t_i)$ for $i = 1, ..., n$. Hence $\sigma_1(l) \to_1 \sigma_1(r)$. Because $\sigma_1(l)$ and $\sigma_1(r)$ are black terms, $\sigma_1(l) \to \sigma_1(r)$ is a rewrite rule of $\mathscr{S}_1$. Therefore $s \equiv C[\sigma_2(\sigma_1(l))] \to_{\mathscr{S}_1} C[\sigma_2(\sigma_1(r))] \equiv t$.

Now we have $\to_{\mathscr{S}_1 \oplus \mathscr{S}_2} = \to_{\mathscr{S}_1} \cup \to_{\mathscr{S}_2} = \to_1 \cup \to_2 = \to_{1,2}$ and hence $\to_{1,2}$ is confluent. ∎

LEMMA 3.7. *If $s \to t$ then $s \downarrow_{1,2} t$.*

Proof. We use induction on the depth of $s \to t$. The case of zero depth is trivial. Suppose the depth of $s \to t$ equals $n+1$ $(n \geqslant 0)$. By definition there exist a context $C[\;]$, a substitution σ, and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_m \downarrow t_m$ in $\mathscr{R}_1 \oplus \mathscr{R}_2$ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$, and $s_i^\sigma \downarrow t_i^\sigma$ for $i = 1, ..., m$ with depth less than or equal to n. Using the induction hypothesis and Lemma 3.6, we obtain $s_i^\sigma \downarrow_{1,2} t_i^\sigma$ for $i = 1, ..., m$; see Fig. 7, where (1) is obtained from the induction hypothesis and (2) signals an application of Lemma 3.6. Without loss of generality we assume that the applied rewrite rule stems from $\mathscr{R}_1$. Proposition 3.13 yields a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $s_i^\tau \downarrow_1^\circ t_i^\tau$ for $i = 1, ..., m$. The next conversion shows that $s \downarrow_{1,2} t$:

$$s \equiv C[l^\sigma] \twoheadrightarrow_{1,2} C[l^\tau] \to_1 C[r^\tau] \twoheadleftarrow_{1,2} C[r^\sigma] \equiv t. \quad \blacksquare$$

Combining Proposition 3.4 and Lemmas 3.6 and 3.7 yields the following result.

PROPOSITION 3.8. *The relations* $=$ *and* $\downarrow_{1,2}$ *coincide.*

Assume $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n$ is a rewrite rule of $\mathscr{R}_1$ and suppose σ is a substitution such that $s_i^\sigma \downarrow_{1,2} t_i^\sigma$ for $i = 1, ..., n$. We have to show the existence of a substitution τ with the properties $\sigma \twoheadrightarrow_{1,2} \tau$ and $s_i^\tau \downarrow_1^\circ t_i^\tau$ for $i = 1, ..., n$. First we show that σ can be transformed into a $\to_{1,2}$-preserved substitution σ', meaning that $\sigma'(x)$ is a $\to_{1,2}$-preserved term for every $x \in \mathscr{D}(\sigma')$.

DEFINITION 3.9. We write $s \to_c t$ if there exist a context $C[\;]$ and terms s_1, t_1 such that $s \equiv C[s_1]$, $t \equiv C[t_1]$, s_1 is a special subterm of s, $s_1 \twoheadrightarrow_{1,2} t_1$, and the root symbols of s_1 and t_1 have different colours. This relation $\to_c$ is called *collapsing reduction* and s_1 is a *collapsing redex*. The relation $\to_c$ is extended to substitutions in the obvious way; i.e., $\sigma \to_c \tau$ if $x^\sigma \to_c x^\tau$ for some $x \in \mathscr{V}$.

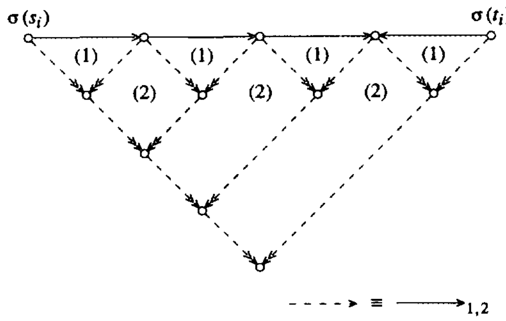

FIGURE 7

PROPOSITION 3.10.

(1) *If* $s \rightarrow_{c} t$ *then* $s \twoheadrightarrow_{1,2} t$.

(2) *A term is* $\rightarrow_{1,2}$*-preserved if and only if it contains no collapsing redexes.*

Proof. Straightforward. ∎

EXAMPLE 3.11. Let

$$\mathscr{R}_1 = \begin{cases} F(x, y) \rightarrow y \Leftarrow x \downarrow G(y) \\ G(x) \quad \rightarrow C \end{cases}$$

and $\mathscr{R}_2 = \{e(x) \rightarrow x\}$. Starting from $t \equiv F(C, e(F(e(C), G(e(C)))))$ we have the following collapsing reduction sequence:

$$\begin{aligned} t &\rightarrow_{c} F(C, e(F(C, G(e(C))))) \\ &\rightarrow_{c} e(F(C, G(e(C)))) \\ &\rightarrow_{c} F(C, G(e(C))) \\ &\rightarrow_{c} F(C, G(C)). \end{aligned}$$

PROPOSITION 3.12. *Collapsing reduction is strongly normalizing.*

Proof. Assign to every term t the multiset $\|t\| = [rank(s) \mid s \in S(t)]$. Suppose that $t \rightarrow_{c} t'$. Using Proposition 2.15, one easily shows that $\|t\| \gg \|t'\|$. Theorem 1.6 yields the strong normalization of $\rightarrow_{c}$ for terms. Combining this with the finiteness of the domain of substitutions yields the strong normalization of $\rightarrow_{c}$ for substitutions. ∎

PROPOSITION 3.13. *Let* $s_1, ..., s_n$, $t_1, ..., t_n$ *be black terms. For every substitution* σ *with* $s_i^{\sigma} \downarrow_{1,2} t_i^{\sigma}$ *for* $i = 1, ..., n$ *there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $s_i^{\tau} \downarrow_1^{\circ} t_i^{\tau}$ *for* $i = 1, ..., n$.

Proof. Let σ' be a normal form of σ with respect to $\rightarrow_{c}$. From Proposition 3.10(1) and Lemma 3.6 we obtain $\sigma'(s_i) \downarrow_{1,2} \sigma'(t_i)$ for $i = 1, ..., n$. Proposition 2.23 yields a decomposition of σ' into $\sigma_2 \circ \sigma_1$ such that σ_1 is black and σ_2 is top white. Note that σ_2 is $\rightarrow_{1,2}$-preserved. Using Proposition 3.14 we obtain a substitution σ^* with $\sigma_2 \twoheadrightarrow_{1,2} \sigma^*$ such that $\sigma^*(\sigma_1(s_i)) \downarrow_1^{\circ} \sigma^*(\sigma_1(t_i))$ for $i = 1, ..., n$. Let τ be the restriction of $\sigma^* \circ \sigma_1$ to $\mathscr{D}(\sigma_1)$. It is easy to show that $\sigma \twoheadrightarrow_{1,2} \tau$. Hence τ satisfies the requirements. ∎

PROPOSITION 3.14. *Let* $s_1, ..., s_n$, $t_1, ..., t_n$ *be black terms. If* σ *is a top* *white and $\rightarrow_{1,2}$-preserved substitution with $s_i^\sigma \downarrow_{1,2} t_i^\sigma$ for $i = 1, \ldots, n$ then there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $s_i^\tau \downarrow_1^\circ t_i^\tau$ for $i = 1, \ldots, n$.*

Proof. According to Proposition 3.15 we can construct a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $x^\sigma \downarrow_{1,2} y^\sigma$ implies $x^\tau \equiv y^\tau$ for all $x, y \in \mathscr{D}(\sigma)$. We show that $s_i^\tau \downarrow_1^\circ t_i^\tau$ for $i = 1, \ldots, n$. Fix i. By definition there exists a term u_i such that $s_i^\sigma \twoheadrightarrow_{1,2} u_i \twoheadleftarrow_{1,2} t_i^\sigma$. Let $A = \{a_1, \ldots, a_m\}$ be the set of all maximal top white subterms occurring in this conversion. We define a mapping ϕ from A to $\{x^\tau \mid x \in \mathscr{D}(\sigma)\}$ as follows:

> Let $a \in A$. From Proposition 3.16 we know that there is a variable $x \in \mathscr{D}(\sigma)$ such that $x^\sigma \twoheadrightarrow_{1,2} a$. We put $\phi(a) \equiv x^\tau$.

We remark that ϕ is well-defined because if there exists another variable $y \in \mathscr{D}(\sigma)$ with $y^\sigma \twoheadrightarrow_{1,2} a$, then $x^\sigma \downarrow_{1,2} y^\sigma$ and hence $x^\tau \equiv y^\tau$. The result of replacing in a term t all maximal special subterms $a \in A$ by the corresponding $\phi(a)$ is denoted by $\Phi(t)$. Let t be any term such that $s_i^\sigma \twoheadrightarrow_{1,2} t$. We prove by induction on the length of the reduction from s_i^σ to t that $\Phi(s_i^\sigma) \twoheadrightarrow_1^\circ \Phi(t)$. If the length is zero then $t \equiv s_i^\sigma$ and we have nothing to prove. Suppose $s_i^\sigma \twoheadrightarrow_{1,2} t' \rightarrow_{1,2} t$. From the induction hypothesis we learn that $\Phi(s_i^\sigma) \twoheadrightarrow_1^\circ \Phi(t')$. By case analysis we show that either $\Phi(t') \equiv \Phi(t)$ or $\Phi(t') \rightarrow_1^\circ \Phi(t)$.

(1) If the rewritten redex in the step $t' \rightarrow_{1,2} t$ occurs in a maximal top white subterm v of t', then we can write $t' \equiv C[v]$ and $t \equiv C[v']$ for some context $C[\ ]$ and term v' with $v \rightarrow_{1,2} v'$. Clearly v and v' (because σ is $\rightarrow_{1,2}$-preserved) are elements of A. Therefore $\phi(v)$ and $\phi(v')$ are defined and since $v \rightarrow_{1,2} v'$, $\phi(v)$ and $\phi(v')$ are identical. We obtain $\Phi(t') \equiv \Phi(t)$.

(2) In the previous case we covered $\rightarrow_1^i$, $\rightarrow_2^i$, and $\rightarrow_2^\circ$ (when $C[\ ] \equiv \Box$). One possibility remains: $t' \rightarrow_1^\circ t$. If t' is a black term (and hence t also is black) then $\Phi(t') \equiv t' \rightarrow_1^\circ t \equiv \Phi(t)$. Otherwise we can write

$$t' \equiv C[\![v_1, \ldots, v_m]\!] \rightarrow_1^\circ C^*\langle\!\langle v_{i_1}, \ldots, v_{i_k} \rangle\!\rangle \equiv t$$

for certain terms $v_1, \ldots, v_m \in A$. Choose pairwise different fresh variables $x_1, \ldots, x_m$ and define terms $w' \equiv C[x_1, \ldots, x_m]$, $w \equiv C^*\langle x_{i_1}, \ldots, x_{i_k} \rangle$, and substitutions $\rho = \{x_i \rightarrow v_i \mid 1 \leqslant i \leqslant m\}$, $\rho' = \{x_i \rightarrow \phi(v_i) \mid 1 \leqslant i \leqslant m\}$. Clearly $\rho \propto \rho'$. Note also that ρ and ρ' are top white. We have $\rho(w') \equiv t' \rightarrow_1^\circ t \equiv \rho(w)$. Proposition 3.5 yields $\rho'(w') \rightarrow_1^\circ \rho'(w)$ and since $\Phi(t') \equiv \rho'(w')$ and $\Phi(t) \equiv \rho'(w)$ we are done.

By the same argument we also have $\Phi(t_i^\sigma) \twoheadrightarrow_1^\circ \Phi(t)$ whenever $t_i^\sigma \twoheadrightarrow_{1,2} t$. Putting everything together, we obtain $s_i^\tau \equiv \Phi(s_i^\sigma) \downarrow_1^\circ \Phi(t_i^\sigma) \equiv t_i^\tau$. ∎

PROPOSITION 3.15. *For every substitution σ there exists a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and if $x^\sigma \downarrow_{1,2} y^\sigma$ then $x^\tau \equiv y^\tau$ for all $x, y \in \mathscr{D}(\sigma)$.*

Proof. Partition the set $\{x^\sigma \mid x \in \mathscr{D}(\sigma)\}$ into equivalence classes $C_1, ..., C_n$ of $\rightarrow_{1,2}$-convertible terms. Because C_i is finite, we may associate with every class C_i a "common reduct" u_i as suggested in Fig. 8. We define the substitution τ by $x^\tau \equiv u_i$ if $x^\sigma \in C_i$ for all $x \in \mathscr{D}(\sigma)$. The substitution τ clearly fulfills the requirements. ∎

PROPOSITION 3.16. *Let t be a black term and suppose σ is a top white and $\rightarrow_{1,2}$-preserved substitution. If $t^\sigma \twoheadrightarrow_{1,2} t'$ and s is a maximal top white subterm of t' then there exists a variable $x \in \mathscr{D}(\sigma)$ such that $x^\sigma \twoheadrightarrow_{1,2} s$.*

Proof. Routine induction on the length of the reduction $t^\sigma \twoheadrightarrow_{1,2} t'$. ∎

THEOREM 3.17. *Confluence is a modular property of join CTRSs.*

Proof. Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ be disjoint join CTRSs. We have to show that $\mathscr{R}_1 \oplus \mathscr{R}_2$ is confluent if and only if both $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are confluent.

$\Rightarrow$ Trivial.

$\Leftarrow$ Consider a conversion $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$. From Proposition 3.8 we obtain $t_1 \downarrow_{1,2} t_2$ and repeated application of Proposition 3.4 yields $t_1 \downarrow t_2$. ∎

The proof of the modularity of confluence for semi-equational CTRSs has exactly the same structure, apart from the proof of Proposition 3.5, which is more complicated because the observation made in order to make the second induction hypothesis applicable is no longer sufficient. In addition to the changed definitions and propositions, we will also give the modified proof of Proposition 3.5. The number of the corresponding definition or proposition for join CTRSs is given in parentheses.

DEFINITION 3.18 (3.2). We write $s \rightarrow_1 t$ if there exist a rewrite rule $l \rightarrow r \Leftarrow s_1 = t_1, ..., s_n = t_n$ in $\mathscr{R}_1$, a context $C[\ ]$, and a substitution σ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$ and $s_i^\sigma =_1^\circ t_i^\sigma$ for $i = 1, ..., n$. The relation $\rightarrow_2$ is defined similarly.

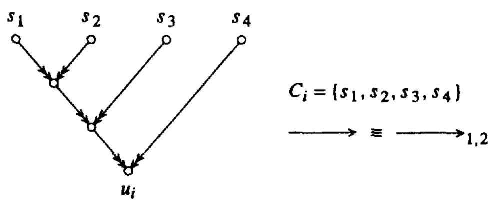

FIGURE 8

PROPOSITION 3.19 (3.5). *Let s, t be black terms and suppose σ is a top white substitution such that $s^\sigma \to_1^\circ t^\sigma$. If τ is a substitution with $\sigma \propto \tau$ then $s^\tau \to_1^\circ t^\tau$.*

Proof. We prove the statement by induction (1) on the depth of $s^\sigma \to_1^\circ t^\sigma$. The case of zero depth is straightforward. If the depth of $s^\sigma \to_1^\circ t^\sigma$ equals $n+1$ ($n \geqslant 0$) then there exist a context $C[\ ]$, a substitution ρ, and a rewrite rule $l \to r \Leftarrow s_1 = t_1, ..., s_m = t_m$ in $\mathscr{R}_1$ such that $s^\sigma \equiv C[\rho(l)]$, $t^\sigma \equiv C[\rho(r)]$, and $\rho(s_i) =_1^\circ \rho(t_i)$ for $i = 1, ..., m$ with depth less than or equal to n. Proposition 2.23 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black, ρ_2 is top white, and $\rho_2 \propto \varepsilon$. We define the substitution ρ^* by $\rho^*(x) \equiv y^\tau$ for every $x \in \mathscr{D}(\rho_2)$ and $y \in \mathscr{D}(\sigma)$ satisfying $\rho_2(x) \equiv y^\sigma$. Note that ρ^* is well-defined by the assumption $\sigma \propto \tau$. We have $\rho_2 \propto \rho^*$ since $\rho_2 \propto \varepsilon$ and $\varepsilon \propto \rho^*$. By induction (2) on the length of the conversion $\rho_2(\rho_1(s_i)) =_1^\circ \rho_2(\rho_1(t_i))$ we will show that $\rho^*(\rho_1(s_i)) =_1^\circ \rho^*(\rho_1(t_i))$ for $i = 1, ..., m$. Fix i. The basis of the induction being trivial, we consider two cases for the induction step.

(1) If $\rho_2(\rho_1(s_i)) \to_1^\circ s' =_1^\circ \rho_2(\rho_1(t_i))$ then we may write

$$\rho_2(\rho_1(s_i)) \equiv C_1\{\{u_1, ..., u_p\}\} \to_1^\circ C_2\langle\!\langle u_{j_1}, ..., u_{j_q}\rangle\!\rangle \equiv s'.$$

For every $u' \in \{u_1, ..., u_p\}$ there is a unique variable $\psi(u') \in \mathscr{D}(\rho_2)$ such that $\rho_2(\psi(u')) \equiv u'$. Hence $s' \equiv \rho_2(s'')$ with $s'' \equiv C_2\langle\psi(u_{j_1}), ..., \psi(u_{j_q})\rangle$ a black term. We obtain $\rho^*(\rho_1(s_i)) \to_1^\circ \rho^*(s'')$ from induction hypothesis (1) and induction hypothesis (2) yields $\rho^*(s'') =_1^\circ \rho^*(\rho_1(t_i))$.

(2) If $\rho_2(\rho_1(s_i)) \leftarrow_1^\circ s' =_1^\circ \rho_2(\rho_1(t_i))$ then we may write

$$\rho_2(\rho_1(s_i)) \equiv C_2\langle\!\langle u_{j_1}, ..., u_{j_q}\rangle\!\rangle \leftarrow_1^\circ C_1\{\{u_1, ..., u_p\}\} \equiv s'.$$

Let $\{v_1, ..., v_r\}$ be the difference between the sets(!) $\{u_1, ..., u_p\}$ and $\{u_{j_1}, ..., u_{j_q}\}$. Choose distinct fresh variables $x_1, ..., x_r$ and define a mapping ψ from $\{u_1, ..., u_p\}$ to $\mathscr{D}(\rho_2) \cup \{x_1, ..., x_r\}$ as follows: if $u' \in \{u_1, ..., u_p\}$ is an element of $\{u_{j_1}, ..., u_{j_q}\}$ then there exists an unique variable $\psi(u') \in \mathscr{D}(\rho_2)$ such that $\rho_2(\psi(u')) \equiv u'$; otherwise $u' \equiv v_k$ for some $k \in \{1, ..., r\}$ and we put $\psi(u') \equiv x_k$. We define the substitution ρ_3 by $\rho_3 = \rho_2 \cup \{x_i \to v_i \mid 1 \leqslant i \leqslant r\}$. By construction we have $\rho_2(\rho_1(s_i)) \equiv \rho_3(\rho_1(s_i))$, $s' \equiv \rho_3(s'')$ with $s'' \equiv C_1\{\psi(u_1), ..., \psi(u_p)\}$ a black term and $\rho_2(\rho_1(t_i)) \equiv \rho_3(\rho_1(t_i))$. Note that ρ_3 is top white and $\rho_3 \propto \rho^*$. Just as in the preceding case, we obtain $\rho^*(\rho_1(s_i)) =_1^\circ \rho^*(\rho_1(t_i))$ from both induction hypotheses.

Hence $\rho^*(\rho_1(l)) \to_1^\circ \rho^*(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing every principal subterm, which has the form x^σ for some variable $x \in \mathscr{D}(\sigma)$, by the corresponding x^τ. A routine argument shows that $s^\tau \equiv C^*[\rho^*(\rho_1(l))]$ and $t^\tau \equiv C^*[\rho^*(\rho_1(r))]$. We conclude that $s^\tau \to_1^\circ t^\tau$. ∎

PROPOSITION 3.20 (3.13). *Let* $s_1, \ldots, s_n$, $t_1, \ldots, t_n$ *be black terms. For every substitution* σ *with* $s_i^\sigma =_{1,2} t_i^\sigma$ $(i = 1, \ldots, n)$ *there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $s_i^\tau =_1^\circ t_i^\tau$ $(i = 1, \ldots, n)$.

PROPOSITION 3.21 (3.14). *Let* $s_1, \ldots, s_n$, $t_1, \ldots, t_n$ *be black terms. If* σ *is a top white and* $\rightarrow_{1,2}$*-preserved substitution with* $s_i^\sigma =_{1,2} t_i^\sigma$ $(i = 1, \ldots, n)$ *then there exists a substitution* τ *such that* $\sigma \twoheadrightarrow_{1,2} \tau$ *and* $s_i^\tau =_1^\circ t_i^\tau$ $(i = 1, \ldots, n)$.

THEOREM 3.22 (3.17). *Confluence is a modular property of semi-equational CTRSs.*

Unlike confluence, local confluence is not a modular property of join CTRSs. This should not come as a surprise since Bergstra and Klop (1986) showed that the Critical Pair Lemma (used in the proof of the modularity of local confluence for TRSs, cf. Middeldorp (1990b)) is not true for join CTRSs.

EXAMPLE 3.23. Let

$$\mathscr{R}_1 = \begin{cases} F(x, y) \rightarrow x \Leftarrow x \downarrow z, z \downarrow y \\ F(x, y) \rightarrow y \Leftarrow x \downarrow z, z \downarrow y \end{cases}$$

and

$$\mathscr{R}_2 = \begin{cases} b \rightarrow a \\ b \rightarrow c \\ c \rightarrow b \\ c \rightarrow d. \end{cases}$$

It is easy to see that $\mathscr{R}_2$ is locally confluent. Let $\mathscr{R}$ be the TRS consisting of the rewrite rule $F(x, x) \rightarrow x$. Clearly $s \rightarrow_{\mathscr{R}} t$ implies $s \rightarrow_{\mathscr{R}_1} t$. Conversely, if $s \rightarrow_{\mathscr{R}_1} t$ then we obtain $s =_{\mathscr{R}} t$ by a straightforward induction on the depth of $s \rightarrow_{\mathscr{R}_1} t$. Because $\mathscr{R}$ is confluent, a routine argument now shows that $\mathscr{R}_1$ is confluent and hence locally confluent. However, $\mathscr{R}_1 \oplus \mathscr{R}_2$ is not locally confluent: we have $a \leftarrow_{\mathscr{R}_1 \oplus \mathscr{R}_2} F(a, d) \rightarrow_{\mathscr{R}_1 \oplus \mathscr{R}_2} d$ since $a \downarrow_{\mathscr{R}_2} b$ and $b \downarrow_{\mathscr{R}_2} d$ and the terms a and d do not have a common reduct.

Because semi-equational CTRSs satisfy the Critical Pair Lemma (Dershowitz, Okada, and Sivakumar, 1988), the refutation of the modularity of local confluence for semi-equational CTRSs is unexpected.

EXAMPLE 3.24. Let

$$\mathscr{R}_1 = \begin{cases} F(x, y) \rightarrow x \Leftarrow x = y \\ F(x, y) \rightarrow y \Leftarrow x = y \end{cases}$$

and let $\mathcal{R}_2$ be the same as in the previous example. We obtain the confluence of $\mathcal{R}_1$ just as in the previous example. The refutation of the local confluence of $\mathcal{R}_1 \oplus \mathcal{R}_2$ is also the same.

4. Strong Normalization

In this section we extend Theorem 2.6 to CTRSs. We show that part (1) of Theorem 2.6 is also true for CTRSs, but for the extension of parts (2) and (3) to CTRSs we have to impose confluence on $\mathcal{R}_1$ and $\mathcal{R}_2$. We first show that strong normalization is a modular property of join CTRSs without collapsing rules. The proof is essentially the same as the one given in Rusinowitch (1987) for TRSs. The only complication is the increased complexity of Proposition 4.3 below.

Notation. We abbreviate $\mathscr{C}(\mathscr{F}_1, \mathscr{V}) \cup \mathscr{C}(\mathscr{F}_2, \mathscr{V})$ to $\mathscr{T}_{\text{top}}$. The restriction of $\rightarrow_{\mathcal{R}_i}$ to $\mathscr{T}_{\text{top}}$ is denoted by $\Rightarrow_i$ $(i = 1, 2)$ and $\Rightarrow$ denotes the union of $\Rightarrow_1$ and $\Rightarrow_2$.

Proposition 4.1. *If $(\mathscr{F}_1, \mathcal{R}_1)$ and $(\mathscr{F}_2, \mathcal{R}_2)$ are disjoint strongly normalizing join CTRSs then $\Rightarrow$ is a strongly normalizing relation.*

Proof. If $\Rightarrow$ is not strongly normalizing then there exists an infinite sequence

$$t_1 \Rightarrow t_2 \Rightarrow t_3 \Rightarrow \cdots.$$

Without loss of generality we assume that $t_1 \in \mathscr{C}(\mathscr{F}_1, \mathscr{V})$. In particular t_1 is in normal form with respect to $\rightarrow_{\mathcal{R}_2}$. Therefore $t_1 \rightarrow_{\mathcal{R}_1} t_2$ and it is easy to see that $t_2 \in \mathscr{C}(\mathscr{F}_1, \mathscr{V})$. Continuing in this way we obtain an infinite reduction sequence

$$t_1 \rightarrow_{\mathcal{R}_1} t_2 \rightarrow_{\mathcal{R}_1} t_3 \rightarrow_{\mathcal{R}_1} \cdots,$$

contradicting the strong normalization of $(\mathscr{F}_1 \cup \{\Box\}, \mathcal{R}_1)$. ∎

Notation. Let σ be a substitution. The substitution $\{x \rightarrow \Box \mid x \in \mathscr{D}(\sigma)\}$ is denoted by $\sigma^{\Box}$.

Until further notice we assume that $(\mathscr{F}_1, \mathcal{R}_1)$ and $(\mathscr{F}_2, \mathcal{R}_2)$ are disjoint strongly normalizing join CTRSs without collapsing rules.

Proposition 4.2. *Let s and t be black terms with $s \notin \mathscr{V}$. If σ is a top white substitution with $s^{\sigma} \rightarrow^{o} t^{\sigma}$ then $\sigma^{\Box}(s) \Rightarrow_1 \sigma^{\Box}(t)$.*

Proof. We use induction on the depth of $s^{\sigma} \rightarrow^{o} t^{\sigma}$. The case of zero depth is straightforward. If the depth of $s^{\sigma} \rightarrow^{o} t^{\sigma}$ equals $n+1$ $(n \geq 0)$ then there exist a context $C[\ ]$, a substitution ρ, and a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_m \downarrow t_m$ in $\mathscr{R}_1$ such that $s^\sigma \equiv C[\rho(l)]$, $t^\sigma \equiv C[\rho(r)]$, and $\rho(s_i) \downarrow \rho(t_i)$ for $i = 1, ..., m$ with depth less than or equal to n. Proposition 2.23 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black and ρ_2 is top white. Fix i. We show the joinability of $\rho_2^{\sqcup}(\rho_1(s_i))$ and $\rho_2^{\sqcup}(\rho_1(t_i))$ with respect to $\Rightarrow_1$ by distinguishing two cases.

(1) Suppose $\rho_1(s_i) \in \mathscr{V}$. If $\rho_1(s_i) \notin \mathscr{D}(\rho_2)$ then $\rho_2(\rho_1(s_i))$ is a variable. Because $\mathscr{R}_1 \oplus \mathscr{R}_2$ contains no collapsing rules, $\rho_2(\rho_1(t_i))$ must be the same variable (otherwise $\rho_2(\rho_1(s_i)) \downarrow \rho_2(\rho_1(t_i))$ cannot be true). Hence

$$\rho_2^{\sqcup}(\rho_1(s_i)) \equiv \rho_1(s_i) \equiv \rho_1(t_i) \equiv \rho_2^{\sqcup}(\rho_1(t_i)).$$

If $\rho_1(s_i) \in \mathscr{D}(\rho_2)$ then $\rho_2(\rho_1(s_i))$ is a top white term and therefore $\rho_2(\rho_1(t_i))$ must also be top white. Hence $\rho_1(t_i) \in \mathscr{D}(\rho_2)$ and $\rho_2^{\sqcup}(\rho_1(s_i)) \equiv \Box \equiv \rho_2^{\sqcup}(\rho_1(t_i))$.

(2) If $\rho_1(s_i) \notin \mathscr{V}$ then $\rho_1(t_i) \notin \mathscr{V}$ by a similar argument as in the previous case. Using the induction hypothesis and considerable effort we obtain the joinability of $\rho_2^{\sqcup}(\rho_1(s_i))$ and $\rho_2^{\sqcup}(\rho_1(t_i))$ with respect to $\Rightarrow_1$ by induction on the length of the valley $\rho_2(\rho_1(s_i)) \downarrow \rho_2(\rho_1(t_i))$.

We have $\rho_2^{\sqcup}(\rho_1(l)) \Rightarrow_1 \rho_2^{\sqcup}(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing all principal subterms by $\Box$. (This is a slight abuse of notation since the resulting context contains in general more than one occurrence of $\Box$.) Because $\sigma^{\sqcup}(s) \equiv C^*[\rho_2^{\sqcup}(\rho_1(l))]$ and $\sigma^{\sqcup}(t) \equiv C^*[\rho_2^{\sqcup}(\rho_1(r))]$ we obtain $\sigma^{\sqcup}(s) \Rightarrow_1 \sigma^{\sqcup}(t)$. ∎

PROPOSITION 4.3. (1) *If* $s \to^{\text{o}} t$ *is not destructive at level* 1 *then* $top(s) \Rightarrow top(t)$.

(2) *If* $s \to^{\text{i}} t$ *is not destructive at level* 2 *then* $top(s) \equiv top(t)$.

Proof. (1) Because there are no collapsing rules, the step $s \to^{\text{o}} t$ is not destructive and according to Proposition 2.14(1) we may write

$$s \equiv C\{\{s_1, ..., s_n\}\} \to^{\text{o}} C^*\{\{s_{i_1}, ..., s_{i_m}\}\} \equiv t.$$

Without loss of generality we assume that s and hence t are top black. Choose distinct fresh variables $x_1, ..., x_n$ and define terms $s' \equiv C\{x_1, ..., x_n\}$ and $t' \equiv C^*\{x_{i_1}, ..., x_{i_m}\}$ and the substitution $\sigma = \{x_i \to s_i \mid 1 \leqslant i \leqslant n\}$. Clearly $s \equiv \sigma(s') \to^{\text{o}} \sigma(t') \equiv t$. Applying Proposition 4.2 yields $\sigma^{\sqcup}(s') \Rightarrow_1 \sigma^{\sqcup}(t')$ and because $\sigma^{\sqcup}(s') \equiv top(s)$ and $\sigma^{\sqcup}(t') \equiv top(t)$ we are done.

(2) We have $s \equiv C[\![s_1, ..., s_j, ..., s_n]\!] \to^{\text{i}} C[\![s_1, ..., t_j, ..., s_n]\!] \equiv t$ with $s_j \to t_j$. Clearly $top(s) \equiv C[\ , ...,\ ] \equiv top(t)$. ∎

DEFINITION 4.4. We define a relation $>_1$ on $\mathcal{T}_\oplus$ as follows: $s >_1 t$ if

(1) $rank(s) \geqslant rank(t)$,

(2) $top(s) \Rightarrow top(t)$ or $top(s) \equiv top(t)$ and $S_2(s) \gg_1 S_2(t)$.

PROPOSITION 4.5. *The relation $>_1$ is strongly normalizing.*

Proof. We will show by induction on $rank(t_1)$ the impossibility of an infinite sequence

$$t_1 >_1 t_2 >_1 t_3 >_1 \cdots.$$

If $rank(t_1) = 1$ then $t_1 \Rightarrow t_2 \Rightarrow t_3 \Rightarrow \cdots$ by definition, contradicting Proposition 4.1. Suppose $rank(t_1) = n$ with $n > 1$. The induction hypothesis states that $>_1$ is strongly normalizing on $\mathcal{T}^i_\oplus$ for all $i < n$. Because $s >_1 t$ implies $rank(s) \geqslant rank(t)$, the relation $>_1$ is also strongly normalizing on $\mathcal{T}^{<n}_\oplus$. Theorem 1.6 yields the strong normalization of $\gg_1$ on $\mathcal{M}(\mathcal{T}^{<n}_\oplus)$. From the definition of $>_1$ and Proposition 4.1 we know that there exists an index i such that

$$S_2(t_i) \gg_1 S_2(t_{i+1}) \gg_1 S_2(t_{i+2}) \gg_1 \cdots.$$

We obtain a contradiction since $S_2(t_j) \in \mathcal{M}(\mathcal{T}^{<n}_\oplus)$ for all $j \geqslant i$. ∎

PROPOSITION 4.6. *If $s \to t$ then $s >_1 t$.*

Proof. Proposition 2.15 yields $rank(s) \geqslant rank(t)$, so we only have to show that $top(s) \Rightarrow top(t)$ or $top(s) \equiv top(t)$ and $S_2(s) \gg_1 S_2(t)$. This will be established by induction on $rank(s)$. If $rank(s) = 1$ then $top(s) \equiv s \Rightarrow t \equiv top(t)$. Let $rank(s) = n$ with $n > 1$. If $s \to^{o} t$ then $top(s) \Rightarrow top(t)$ by Proposition 4.3(1). If $s \to^{i} t$ then $top(s) \equiv top(t)$ by Proposition 4.3(2) and we may write $s \equiv C[\![s_1, ..., s_j, ..., s_m]\!] \to C[\![s_1, ..., t_j, ..., s_m]\!] \equiv t$ with $s_j \to t_j$. The induction hypothesis yields $s_j >_1 t_j$. Hence

$$S_2(s) = [s_1, ..., s_j, ..., s_m] \gg_1 [s_1, ..., t_j, ..., s_m] = S_2(t). \quad ∎$$

THEOREM 4.7. *Strong normalization is a modular property of join CTRSs without collapsing rules.*

Proof. Immediate consequence of Propositions 4.5 and 4.6. ∎

Surprisingly, parts (2) and (3) of Theorem 2.6 are not true for join CTRSs. The following example refutes both statements.

EXAMPLE 4.8. Let $\mathscr{R}_1 = \{F(x) \to F(x) \Leftarrow x \downarrow A, x \downarrow B\}$ and

$$\mathscr{R}_2 = \begin{cases} or(x, y) \to x \\ or(x, y) \to y. \end{cases}$$

Clearly $\to_{\mathscr{R}_1}$ coincides with the empty relation and therefore $\mathscr{R}_1$ is strongly normalizing. The strong normalization of $\mathscr{R}_2$ is obvious. In $\mathscr{R}_1 \oplus \mathscr{R}_2$ the term $F(or(A, B))$ reduces to itself since $or(A, B) \downarrow_{\mathscr{R}_2} A$ and $or(A, B) \downarrow_{\mathscr{R}_2} B$. Note that both systems do not contain duplicating rules. Furthermore, $\mathscr{R}_1$ lacks collapsing rules and $\mathscr{R}_2$ is not confluent.

We proceed by showing that parts (2) and (3) of Theorem 2.6 are true for join CTRSs under the additional requirement of confluence. The following two propositions are illustrated in Figs. 9 and 10.

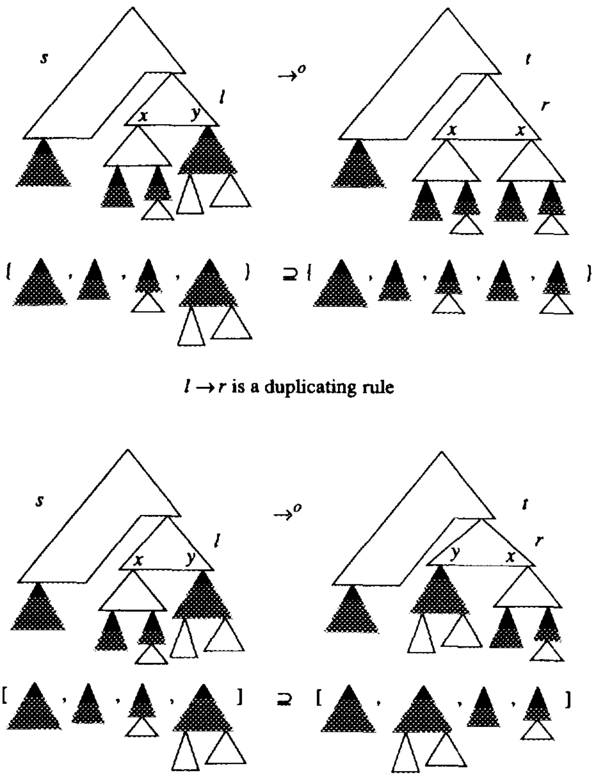

FIGURE 9

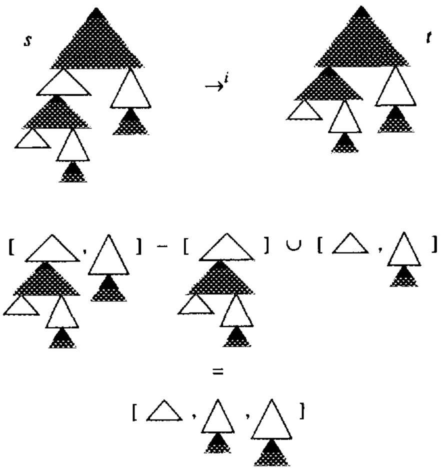

FIGURE 10

PROPOSITION 4.9. *If $s \to^{o} t$ is a non-destructive rewrite step then the set inclusion $\{u \mid u \in S_2(t)\} \subseteq \{u \mid u \in S_2(s)\}$ holds. If the applied rewrite rule is not duplicating, we even have the multiset inclusion $S_2(t) \subseteq S_2(s)$.*

Proof. Straightforward. ▮

PROPOSITION 4.10. *If $s \equiv C[\![s_1, ..., s_j, ..., s_n]\!] \to^{i} C[s_1, ..., t_j, ..., s_n] \equiv t$ is destructive at level 2 then $S_2(t) = S_2(s) - [s_j] \cup S_2(t_j)$.*

Proof. Routine. ▮

The proofs of parts (2) and (3) of Theorem 2.6 given in Rusinowitch (1987) and Middeldorp (1989b) use the observation that $top(s) \Rightarrow top(t)$ whenever $s \to^{o} t$ is non-destructive. The next example shows that in the presence of collapsing rules this observation is no longer true for join CTRSs, even if they are confluent.

EXAMPLE 4.11. Let $\mathscr{R}_1 = \{F(x) \to F(A) \Leftarrow x \downarrow B\}$ and $\mathscr{R}_2 = \{e(x) \to x\}$. The rewrite step $F(e(B)) \to^{o} F(A)$ is not destructive but clearly $top(F(e(B))) \equiv F(\Box)$ is not $\Rightarrow$-reducible.

We now show that the observation "$top(s) \Rightarrow top(t)$ whenever $s \to^{o} t$ is non-destructive" can be retrieved by adding to $top(t)$ some of the information concealed in the inner parts of t, provided the participating CTRSs are confluent and strongly normalizing. So assume that $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint complete join CTRSs.

PROPOSITION 4.12. *The relation $\rightarrow_{1,2}$ is weakly normalizing.*

Proof. As in the proof of Lemma 3.6 we define TRSs $(\mathscr{F}_1, \mathscr{S}_1)$ and $(\mathscr{F}_2, \mathscr{S}_2)$ by $(i = 1, 2)$

$$\mathscr{S}_i = \{s \rightarrow t \mid s, t \in \mathscr{T}_i \text{ and } s \rightarrow_i t\}.$$

Because the restrictions of $\rightarrow_{\mathscr{S}_i}$, $\rightarrow_i$, and $\rightarrow_{\mathscr{R}_i}$ to $\mathscr{T}_i \times \mathscr{T}_i$ are the same, both TRSs are strongly normalizing and hence also weakly normalizing. Theorem 2.7 yields the weak normalization of $\mathscr{S}_1 \oplus \mathscr{S}_2$. In the proof of Lemma 3.6 we already observed that the relations $\rightarrow_{\mathscr{S}_1 \oplus \mathscr{S}_2}$ and $\rightarrow_{1,2}$ coincide. Therefore $\rightarrow_{1,2}$ is weakly normalizing. ∎

Because $\rightarrow_{1,2}$ is also confluent (Lemma 3.6), every term t has a unique normal form with respect to $\rightarrow_{1,2}$. This normal form is denoted by $t^{\rightarrow}$.

DEFINITION 4.13. Let $t \in \mathscr{T}_{\oplus}$. We define $top^{\rightarrow}(t)$ as follows:

$$top^{\rightarrow}(t) = \begin{cases} t & \text{if } rank(t) = 1, \\ top(C[t_1^{\rightarrow}, ..., t_n^{\rightarrow}]) & \text{if } t \equiv C[\![t_1, ..., t_n]\!]. \end{cases}$$

EXAMPLE 4.14. Consider again the CTRSs of Example 4.11. We have

$$top^{\rightarrow}(F(e(B)) \equiv F(B) \Rightarrow F(A) \equiv top^{\rightarrow}(F(A)).$$

PROPOSITION 4.15. *If s and t are black terms and σ is a top white $\rightarrow_{1,2}$-normalized substitution such that $s^{\sigma} \downarrow_{1,2} t^{\sigma}$, then $s^{\sigma} \downarrow_1^{\circ} t^{\sigma}$.*

Proof. We use induction on the length of the valley $s^{\sigma} \downarrow_{1,2} t^{\sigma}$. The case of zero length is trivial. Let $s^{\sigma} \rightarrow_{1,2} s_1 \downarrow_{1,2} t^{\sigma}$. (The case $s^{\sigma} \downarrow_{1,2} t_1 \leftarrow_{1,2} t^{\sigma}$ is similar.) Because σ is top white $\rightarrow_{1,2}$-normalized and s is a black term, this implies that $s \notin \mathscr{V}$ and $s^{\sigma} \rightarrow_1^{\circ} s_1$. It is not difficult to see that there exists a black term s_2 such that $s_1 \equiv s_2^{\sigma}$. The induction hypothesis yields $s_2^{\sigma} \downarrow_1^{\circ} t^{\sigma}$ and thus we have $s^{\sigma} \downarrow_1^{\circ} t^{\sigma}$. ∎

Notation. Let σ be a substitution. The substitution $\{x \rightarrow \sigma(x)^{\rightarrow} \mid x \in \mathscr{D}(\sigma)\}$ is denoted by $\sigma^{\rightarrow}$. Clearly $\sigma \twoheadrightarrow_{1,2} \sigma^{\rightarrow}$.

PROPOSITION 4.16. *Let s and t be black terms with $s \notin \mathscr{V}$. If σ is a top white substitution such that $s^{\sigma} \rightarrow^{\circ} t^{\sigma}$ then $\sigma^{\rightarrow}(s) \rightarrow_1^{\circ} \sigma^{\rightarrow}(t)$.*

Proof. There exist a context $C[\ ]$, a substitution ρ, and a rewrite rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n$ $(n \geqslant 0)$ in $\mathscr{R}_1$ such that $s^{\sigma} \equiv C[\rho(l)]$, $t^{\sigma} \equiv C[\rho(r)]$, and $\rho(s_i) \downarrow \rho(t_i)$ for $i = 1, ..., n$. Proposition 2.23 yields a decomposition $\rho_2 \circ \rho_1$ of ρ such that ρ_1 is black and ρ_2 is top white. Fix i. We show that $\rho_2^{\rightarrow}(\rho_1(s_i)) \downarrow_1^{\circ} \rho_2^{\rightarrow}(\rho_1(t_i))$. From Proposition 3.8 we obtain $\rho_2(\rho_1(s_i)) \downarrow_{1,2}$

$\rho_2(\rho_1(t_i))$. Because $\rho_2 \twoheadrightarrow_{1,2} \rho_2^{\rightarrow}$, an application of Lemma 3.6 yields $\rho_2^{\rightarrow}(\rho_1(s_i)) \downarrow_{1,2} \rho_2^{\rightarrow}(\rho_1(t_i))$. According to Proposition 2.23 we may decompose $\rho_2^{\rightarrow}$ into $\rho_4 \circ \rho_3$ such that ρ_3 is black and ρ_4 is top white. Note that ρ_4 is $\rightarrow_{1,2}$-normalized. Proposition 4.15 yields $\rho_4(\rho_3(\rho_1(s_i))) \downarrow_1^{\circ} \rho_4(\rho_3(\rho_1(t_i)))$. We have $\rho_2^{\rightarrow}(\rho_1(l)) \rightarrow_1^{\circ} \rho_2^{\rightarrow}(\rho_1(r))$. Let $C^*[\ ]$ be the context obtained from $C[\ ]$ by replacing all principal subterms by their respective $\rightarrow_{1,2}$-normal forms. Clearly $\sigma^{\rightarrow}(s) \equiv C^*[\rho_2^{\rightarrow}(\rho_1(l))]$ and $\sigma^{\rightarrow}(t) \equiv C^*[\rho_2^{\rightarrow}(\rho_1(r))]$. We conclude that $\sigma^{\rightarrow}(s) \rightarrow_1^{\circ} \sigma^{\rightarrow}(t)$. ∎

PROPOSITION 4.17. (1) *If* $s \rightarrow^{\circ} t$ *is not destructive at level* 1 *then* $top^{\rightarrow}(s) \Rightarrow top^{\rightarrow}(t)$.

(2) *If* $s \rightarrow^{\mathrm{i}} t$ *is not destructive at level* 2 *then* $top^{\rightarrow}(s) \equiv top^{\rightarrow}(t)$.

Proof. (1) According to Proposition 2.14(1) we may write $s \equiv C\{\{s_1, ..., s_n\}\}$ and $t \equiv C^*\{\{s_{i_1}, ..., s_{i_m}\}\}$. Without loss of generality we assume that s and hence t are top black. Let $x_1, ..., x_n$ be distinct fresh variables and define the substitution $\sigma = \{x_i \rightarrow s_i \mid 1 \leqslant i \leqslant n\}$ and terms $s' \equiv C\{x_1, ..., x_n\}$ and $t' \equiv C^*\{x_{i_1}, ..., x_{i_m}\}$. Because σ is top white we can apply Proposition 4.16. This gives us $\sigma^{\rightarrow}(s') \rightarrow_1^{\circ} \sigma^{\rightarrow}(t')$. Proposition 2.23 yields a decomposition $\sigma_2 \circ \sigma_1$ of $\sigma^{\rightarrow}$ such that σ_1 is black and σ_2 is top white. Since $\sigma_2 \propto \sigma_2^{\sqcup}$ we obtain $\sigma_2^{\sqcup}(\sigma_1(s')) \rightarrow_1^{\circ} \sigma_2^{\sqcup}(\sigma_1(t'))$ from Proposition 3.5. It is easy to see that $\sigma_2^{\sqcup}(\sigma_1(s')) \Rightarrow \sigma_2^{\sqcup}(\sigma_1(t'))$. We have

$$top^{\rightarrow}(s) \equiv top(C\{s_1^{\rightarrow}, ..., s_n^{\rightarrow}\}) \equiv top(\sigma^{\rightarrow}(s')) \equiv top(\sigma_2(\sigma_1(s'))) \equiv \sigma_2^{\sqcup}(\sigma_1(s')),$$

where the last identity follows from the fact that $\sigma_1(s') \notin \mathscr{V}$. Similarly $top^{\rightarrow}(t) \equiv \sigma_2^{\sqcup}(\sigma_1(t'))$ and therefore $top^{\rightarrow}(s) \Rightarrow top^{\rightarrow}(t)$.

(2) We have $s \equiv C[\![s_1, ..., s_j, ..., s_n]\!] \rightarrow^{\mathrm{i}} C[\![s_1, ..., t_j, ..., s_n]\!] \equiv t$ with $s_j \rightarrow t_j$. Lemma 3.7 yields $s_j \downarrow_{1,2} t_j$ and hence s_j and t_j have the same $\rightarrow_{1,2}$-normal form. Therefore

$$\begin{aligned} top^{\rightarrow}(s) &\equiv top(C[s_1^{\rightarrow}, ..., s_j^{\rightarrow}, ..., s_n^{\rightarrow}]) \\ &\equiv top(C[s_1^{\rightarrow}, ..., t_j^{\rightarrow}, ..., s_n^{\rightarrow}]) \equiv top^{\rightarrow}(t). \quad ∎ \end{aligned}$$

With the above results in hand we can easily modify the proofs of parts (2) and (3) of Theorem 2.6 given in Rusinowitch (1987) and Middeldorp (1989b). First assume that $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint complete join CTRSs without duplicating rules.

DEFINITION 4.18. Let $t \in \mathscr{T}_{\oplus}$. We define $\# t$ as the cardinality of the multiset $S(t)$, provided t is not a variable. If $t \in \mathscr{V}$ then $\# t = 0$.

Note that $\# t$ denotes the number of black and white parts in t. The special treatment of variables allows a more concise formulation of the proof of Proposition 4.21 below.

Notation. The multiset $[top^{\rightarrow}(s) \mid s \in S(t)]$ is denoted by $\Delta(t)$.

DEFINITION 4.19. We define a relation $>_2$ on $\mathscr{T}_{\oplus}$ as follows: $s >_2 t$ if $\#s > \#t$ or $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$.

PROPOSITION 4.20. *The relation $>_2$ is strongly normalizing.*

Proof. Suppose $>_2$ is not strongly normalizing. It is easy to show that there exists an infinite sequence

$$t_1 >_2 t_2 >_2 t_3 >_2 \cdots$$

in which all terms have the same number of black and white parts. Hence we have the infinite sequence

$$\Delta(t_1) \Rightarrow^m \Delta(t_2) \Rightarrow^m \Delta(t_3) \Rightarrow^m \cdots.$$

But this is impossible, since combining Proposition 4.1 and Theorem 1.6 yields the strong normalization of $\Rightarrow^m$. ▮

PROPOSITION 4.21. *If $s \rightarrow t$ then $s >_2 t$.*

Proof. We will show by induction on $rank(s)$ that either $\#s > \#t$ or $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$. First assume that $rank(s) = 1$. If $s \rightarrow t$ is destructive then $\#s = 1 > 0 = \#t$. Otherwise $\#s = \#t = 1$ and $top(s) \equiv s \Rightarrow t \equiv top(t)$. Now let $rank(s) = n$ with $n > 1$. We distinguish two cases.

(1) If $s \rightarrow^{\circ} t$ is destructive then either $t \in V(top(s))$ or $t \in S_2(s)$. In both cases we clearly have $\#s > \#t$. If $s \rightarrow^{\circ} t$ is not destructive then $S_2(t) \subseteq S_2(s)$ by Proposition 4.9 and therefore $S_i(t) \subseteq S_i(s)$ for all $i \geqslant 2$. Proposition 4.17(1) yields $top^{\rightarrow}(s) \Rightarrow top^{\rightarrow}(t)$. Hence

$$\begin{aligned} \Delta(s) &= [top^{\rightarrow}(s)] \cup [top^{\rightarrow}(u) \mid u \in S_{>1}(s)] \\ &\Rightarrow^m [top^{\rightarrow}(t)] \cup [top^{\rightarrow}(u) \mid u \in S_{>1}(t)] = \Delta(t). \end{aligned}$$

(2) If $s \rightarrow^{i} t$ is destructive at level 2 then we easily obtain $\#s > \#t$. Otherwise we may write $s \equiv C[\![s_1, ..., s_j, ..., s_m]\!] \rightarrow^{i} C[\![s_1, ..., t_j, ..., s_m]\!] \equiv t$ with $s_j \rightarrow t_j$. The induction hypothesis yields $s_j >_2 t_j$. If $\#s_j > \#t_j$ then $\#s > \#t$. If $\#s_j = \#t_j$ and $\Delta(s_j) \Rightarrow^m \Delta(t_j)$ then also $\#s = \#t$ and $\Delta(s) \Rightarrow^m \Delta(t)$. ▮

THEOREM 4.22. *Strong normalization is a modular property of confluent join CTRSs without duplicating rules.*

Proof. Immediate consequence of Propositions 4.20 and 4.21. ∎

Finally we consider the case where $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are complete join CTRSs such that one of them contains neither collapsing nor duplicating rules. Without loss of generality we assume that $(\mathscr{F}_1, \mathscr{R}_1)$ contains neither collapsing nor duplicating rules. Our proof can be seen as an extension of Theorem 4.7. We refine the relation $>_1$ of Definition 4.4 by associating with every term a quantity which decreases when that term is reduced by a destructive rewrite step at level 1 or 2, and does not increase otherwise.

DEFINITION 4.23. To each term $t \in \mathscr{T}_{\oplus}$ we assign a weight $\|t\|$ as follows:

$$\|t\| = \begin{cases} 0 & \text{if } t \in \mathscr{V}, \\ \sum_{s \in S_2(t)} \|s\| & \text{if } t \text{ is top black,} \\ 1 + \max_{s \in S_2(t)} \|s\| & \text{if } t \text{ is top white.} \end{cases}$$

EXAMPLE 4.24. Let

$$\mathscr{R}_1 = \begin{cases} F(x, y, z) \to G(z) & \Leftarrow x \downarrow y \\ G(A) \to F(A, B, A) \end{cases}$$

and

$$\mathscr{R}_2 = \begin{cases} e(x) \to f(x, x) \\ f(x, y) \to x. \end{cases}$$

In the reduction sequence

$$\begin{aligned} & e(F(f(G(A), B), G(A), e(B))) \\ & \quad \to f(F(f(G(A), B), G(A), e(B)), F(f(G(A), B), G(A), e(B))) \\ & \quad \to f(F(G(A), G(A), e(B)), F(f(G(A), B), G(A), e(B))) \\ & \quad \to F(G(A), G(A), e(B)) \\ & \quad \to G(e(B)) \\ & \quad \to G(f(B, B)) \\ & \quad \to G(B) \end{aligned}$$

we have the weights 3, 3, 3, 1, 1, 1, and 0, respectively.

PROPOSITION 4.25. *If $s \to t$ is destructive at level 1 then $\|s\| > \|t\|$.*

Proof. We either have $s \equiv C[\![s_1, ..., s_n]\!] \to s_i \equiv t$ or $s \to x \equiv t$ for some variable $x \in V(top(s))$. In the former case we obtain

$$\|s\| = 1 + \max\{\|s_j\| \mid 1 \leqslant j \leqslant n\} > \|s_i\| = \|t\|$$

because s is top white and in the latter case we clearly have $\|s\| > 0 = \|t\|$. ∎

PROPOSITION 4.26. *If $s \to t$ is destructive at level 2 then $\|s\| > \|t\|$.*

Proof. We have $\equiv C[\![s_1, ..., s_j, ..., s_n]\!] \to C[s_1, ..., t_j, ..., s_n] \equiv t$ with $s_j \to t_j$ destructive at level 1. From Proposition 4.25 we obtain $\|s_j\| > \|t_j\|$. Note that s and t are top black. Hence

$$\|s\| = \sum_{i=1}^{n} \|s_i\|$$

and

$$\|t\| = \|s\| - \|s_j\| + \sum_{u \in S_2(t_j)} \|u\|$$

by Proposition 4.10. We only have to show that

$$\|s_j\| > \sum_{u \in S_2(t_j)} \|u\|.$$

Because $s_j \to t_j$ is destructive at level 1, we either have $t_j \in V(top(s_j))$ or $t_j \in S_2(s_j)$. In the first case we clearly have

$$\|s_j\| > 0 = \sum_{u \in [\ ]} \|u\|$$

and in the second case we obtain

$$\|s_j\| > \|t_j\| = \sum_{u \in S_2(t_j)} \|u\|$$

since t_j is top black. ∎

The second step in the reduction sequence of Example 4.24 shows that the previous propositions do not generalize to destructive rewrite steps at a level greater than 2.

PROPOSITION 4.27. *If $s \to t$ then $\|s\| \geqslant \|t\|$.*

Proof. Using Propositions 4.25 and 4.26 we may assume that $s \to t$ is not destructive at level 1 or 2. We will use induction on $rank(s)$. If $rank(s) = 1$ then $rank(t) = 1$ by Proposition 2.15. We have $\|s\| = 0 = \|t\|$ if s and t are top black and because t is not a variable (otherwise $s \to t$ would be destructive at level 1) we have $\|s\| = 1 = \|t\|$ if s and t are top white. Assume the statement is true for all terms with rank less than n ($n > 1$) and let $rank(s) = n$. We distinguish two cases.

(1) If $s \to^{o} t$ then $\{u \mid u \in S_2(t)\} \subseteq \{u \mid u \in S_2(s)\}$ by Proposition 4.9. If the applied rewrite rule is duplicating then s and t are top white and

$$\|s\| = 1 + \max_{u \in S_2(s)} \|u\| \geqslant 1 + \max_{u \in S_2(t)} \|u\| = \|t\|.$$

If the applied rewrite rule is not duplicating, we obtain the multiset inclusion $S_2(t) \subseteq S_2(s)$ from Proposition 4.9. Therefore we have both

$$\sum_{u \in S_2(s)} \|u\| \geqslant \sum_{u \in S_2(t)} \|u\|$$

and

$$1 + \max_{u \in S_2(s)} \|u\| \geqslant 1 + \max_{u \in S_2(t)} \|u\|,$$

so we always have $\|s\| \geqslant \|t\|$.

(2) If $s \to^{i} t$ then $s \equiv C[\![s_1, \ldots, s_j, \ldots, s_m]\!] \to C[\![s_1, \ldots, t_j, \ldots, s_m]\!] \equiv t$ with $s_j \to t_j$. The induction hypothesis yields $\|s_j\| \geqslant \|t_j\|$. Clearly $S_2(t) = S_2(s) - [s_j] \cup [t_j]$. So again we have both

$$\sum_{u \in S_2(s)} \|u\| \geqslant \sum_{u \in S_2(t)} \|u\|$$

and

$$1 + \max_{u \in S_2(s)} \|u\| \geqslant 1 + \max_{u \in S_2(t)} \|u\|.$$

Hence $\|s\| \geqslant \|t\|$. ∎

DEFINITION 4.28. We define a relation $>_3$ on $\mathscr{T}_{\oplus}$ as follows: $s >_3 t$ if

(1) $rank(s) \geqslant rank(t)$,

(2) $\|s\| > \|t\|$ or $\|s\| = \|t\|$ and $top^{\rightarrow}(s) \Rightarrow top^{\rightarrow}(t)$ or $\|s\| = \|t\|$, $top^{\rightarrow}(s) \equiv top^{\rightarrow}(t)$ and $S_2(s) \gg_3 S_2(t)$.

PROPOSITION 4.29. *The relation $>_3$ is strongly normalizing.*

Proof. Similar to the proof of Proposition 4.5. ∎

PROPOSITION 4.30. *If* $s \to t$ *then* $s >_3 t$.

Proof. Since $rank(s) \geqslant rank(t)$ by Proposition 2.15, we only have to show that $\|s\| > \|t\|$ or $\|s\| = \|t\|$ and $top^{\to}(s) \Rightarrow top^{\to}(t)$ or $\|s\| = \|t\|$, $top^{\to}(s) \equiv top^{\to}(t)$, and $S_2(s) \gg_3 S_2(t)$. This will be done using induction on $rank(s)$. First we consider the case $rank(s) = 1$. If $s \to t$ is destructive at level 1 then $\|s\| > \|t\|$ by Proposition 4.25. Otherwise $\|s\| = \|t\|$ and $top^{\to}(s) \Rightarrow top^{\to}(t)$ by Proposition 4.17(1). We now assume that $rank(s) = n$ with $n > 1$. Proposition 4.27 yields $\|s\| \geqslant \|t\|$. We distinguish two cases.

(1) If $s \to^{o} t$ is destructive at level 1 then $\|s\| > \|t\|$ by Proposition 4.25 and if $s \to^{o} t$ is not destructive then $top^{\to}(s) \Rightarrow top^{\to}(t)$ by Proposition 4.17(1).

(2) If $s \to^{i} t$ is destructive at level 2 then the result follows from Proposition 4.26. If $s \to^{i} t$ is not destructive at level 2 then $top^{\to}(s) \equiv top^{\to}(t)$ by Proposition 4.17(2) and we may write

$$s \equiv C[\![s_1, ..., s_j, ..., s_m]\!] \to C[\![s_1, ..., t_j, ..., s_m]\!] \equiv t$$

with $s_j \to t_j$. From the induction hypothesis we obtain $s_j >_3 t_j$. Therefore

$$S_2(s) = [s_1, ..., s_j, ..., s_m] \gg_3 [s_1, ..., t_j, ..., s_m] = S_2(t).$$ ∎

THEOREM 4.31. *If* $(\mathscr{F}_1, \mathscr{R}_1)$ *and* $(\mathscr{F}_2, \mathscr{R}_2)$ *are disjoint complete join CTRSs such that one of them contains neither collapsing nor duplicating rules, then* $\mathscr{R}_1 \oplus \mathscr{R}_2$ *is strongly normalizing.*

Proof. Immediate consequence of Propositions 4.29 and 4.30. ∎

For semi-equational CTRSs the situation is the same: part (1) of Theorem 4.6 holds but parts (2) and (3) require confluence. The next example is a slight simplification of the corresponding one for join CTRSs.

EXAMPLE 4.32. Let $\mathscr{R}_1 = \{F(x) \to F(A) \Leftarrow x = B\}$ and

$$\mathscr{R}_2 = \begin{cases} or(x, y) \to x \\ or(x, y) \to y. \end{cases}$$

Both CTRSs are strongly normalizing and $F(A) \to_{\mathscr{R}_1 \oplus \mathscr{R}_2} F(A)$ because $A \leftarrow_{\mathscr{R}_2} or(A, B) \to_{\mathscr{R}_2} B$.

THEOREM 4.33. *Let* $(\mathscr{F}_1, \mathscr{R}_1)$ *and* $(\mathscr{F}_2, \mathscr{R}_2)$ *be disjoint strongly normalizing semi-equational CTRSs.*

(1) *If both systems do not contain collapsing rules then* $\mathscr{R}_1 \oplus \mathscr{R}_2$ *is strongly normalizing.*

(2) *If both systems are confluent and do not contain duplicating rules then* $\mathscr{R}_1 \oplus \mathscr{R}_2$ *is strongly normalizing.*

(3) *If both systems are confluent and one of them contains neither collapsing nor duplicating rules then* $\mathscr{R}_1 \oplus \mathscr{R}_2$ *is strongly normalizing.*

5. Weak Normalization

In contrast to strong normalization, weak normalization is a modular property of TRSs. This has been independently observed by several authors (Bergstra, Klop, and Middeldorp, 1989; Drosten, 1989; Kurihara and Kaji, 1990; Toyama, Klop, and Barendregt, 1989). Two approaches can be identified in establishing the weak normalization of the disjoint union $\mathscr{R}_1 \oplus \mathscr{R}_2$ of two weakly normalizing TRSs $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$:

(1) Every term $t \in \mathscr{T}_\oplus$ can be normalized using "innermost" rewriting; i.e., first the bottom layer of t is reduced to normal form, then the layer above the bottom layer is normalized, and working steadily upwards we eventually normalize t.

(2) A term $t \in \mathscr{T}_\oplus$ can also be normalized by the following recipe: First we normalize t with respect to $\mathscr{R}_1$ with result, say, t_1. The term t_1 is then normalized with respect to $\mathscr{R}_2$ giving t_2. Now we again use $\mathscr{R}_1$ to normalize t_2 and continuing in this manner we eventually arrive at an $\mathscr{R}_1 \oplus \mathscr{R}_2$-normal form of t. The termination of this process is guaranteed by an interesting result of Kurihara and Kaji (1990).

Both methods rely on the equality of $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ and $NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$, which is a consequence of the equality of $\rightarrow_{\mathscr{R}_1 \oplus \mathscr{R}_2}$ and $\rightarrow_{\mathscr{R}_1} \cup \rightarrow_{\mathscr{R}_2}$. In Section 3 we observed that this equality does not hold for CTRSs. The following example shows that weak normalization is not a modular property of join CTRSs.

Example 5.1. Let

$$\mathscr{R}_1 = \begin{cases} F(x, x) \rightarrow C \\ F(x, y) \rightarrow F(x, y) \Leftarrow x \downarrow z, z \downarrow y \end{cases}$$

and

$$\mathscr{R}_2 = \begin{cases} a \rightarrow b \\ a \rightarrow c. \end{cases}$$

One easily shows that $\mathscr{R}_1$ is confluent. From this we obtain the weak normalization of $\mathscr{R}_1$ by a routine argument. Clearly $\mathscr{R}_2$ is weakly normalizing. However, $\mathscr{R}_1 \oplus \mathscr{R}_2$ is not weakly normalizing: the term $F(b, c)$ reduces only to itself. Note that the rewrite rule of $\mathscr{R}_1$ contains an extra variable (z) in the conditions and $\mathscr{R}_2$ is not confluent.

The proof of the next theorem is based on method (1) for proving the modularity of weak normalization for TRSs. A proof based on method (2) is also possible (see Middeldorp (1990b) for details).

THEOREM 5.2. *If $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint weakly normalizing join CTRSs such that $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$, then $\mathscr{R}_1 \oplus \mathscr{R}_2$ is weakly normalizing.*

Proof. We show by induction on $rank(t)$ that every term t has a normal form with respect to $\mathscr{R}_1 \oplus \mathscr{R}_2$. If $rank(t) = 1$ then the result follows from the assumption that $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are weakly normalizing. Let $t \equiv C[\![t_1, ..., t_n]\!]$. Without loss of generality we assume that t is top black. Applying the induction hypothesis to $t_1, ..., t_n$ yields normal forms $t'_1, ..., t'_n$ such that $t_i \twoheadrightarrow t'_i$ for $i = 1, ..., n$. We clearly have $C[t'_1, ..., t'_n] \equiv C'\{\{s_1, ..., s_m\}\}$ for some context $C'\{\,, ..., \}$ and top white normal forms $s_1, ..., s_m$. Choose fresh variables $x_1, ..., x_m$ with $\langle s_1, ..., s_m\rangle \propto \langle x_1, ..., x_m\rangle$. Because $rank(C'\{x_1, ..., x_m\}) = 1$, the term $C'\{x_1, ..., x_m\}$ has a normal form, say

$$C'\{x_1, ..., x_m\} \twoheadrightarrow_{\mathscr{R}_1} C^*\langle x_{i_1}, ..., x_{i_p}\rangle.$$

Hence we have the following reduction sequence:

$$t \twoheadrightarrow_{\mathscr{R}_1 \oplus \mathscr{R}_2} C'\{\{s_1, ..., s_m\}\} \twoheadrightarrow^{\circ}_{\mathscr{R}_1} C^*\langle\!\langle s_{i_1}, ..., s_{i_p}\rangle\!\rangle \equiv t'.$$

Clearly $t' \in NF(\mathscr{F}_\oplus, \mathscr{R}_2)$. By construction we have $t' \in NF(\rightarrow^{\circ}_{\mathscr{R}_1})$ and since $s_{i_1}, ..., s_{i_p} \in NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ we also have $t' \in NF(\mathscr{F}_\oplus, \mathscr{R}_1)$. The assumption $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ yields $t' \in NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$. We conclude that every term has a normal form with respect to $\mathscr{R}_1 \oplus \mathscr{R}_2$. ∎

Example 5.1 suggests two sufficient conditions for the equality of $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ and $NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$, and hence for the modularity of weak normalization for join CTRSs.

PROPOSITION 5.3. *If $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint join CTRSs without extra variables in the conditions then $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$.*

Proof. ($\subseteq$) Trivial.

($\supseteq$) If $NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ is not a subset of $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ then there exists a smallest term t such that $t \in NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ and $t \notin NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$. Clearly t must be a redex, so there is a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n$ in $\mathscr{R}_1 \oplus \mathscr{R}_2$ and a substitution σ such that $t \equiv l^\sigma$ and $s_i^\sigma \downarrow t_i^\sigma$ for $i = 1, ..., n$. Assume without loss of generality that the rewrite rule stems from $\mathscr{R}_1$. Because $V(u) \subseteq V(l)$ for all $u \in \{s_1, ..., s_n, t_1, ..., t_n\}$ we may assume that $\mathscr{D}(\sigma) \subseteq V(l)$. Due to the minimality of t, $x^\sigma \in NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ for every $x \in \mathscr{D}(\sigma)$. Using this fact, we can easily show that $s_i^\sigma \downarrow_{\mathscr{R}_1} t_i^\sigma$ for $i = 1, ..., n$. But then $l^\sigma \to_{\mathscr{R}_1} r^\sigma$, contradicting the assumption $t \in NF(\mathscr{F}_\oplus, \mathscr{R}_1)$. ∎

COROLLARY 5.4. *Weak normalization is a modular property of join CTRSs without extra variables in the conditions of the rewrite rules.*

The sufficiency of confluence for the equality of $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ and $NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ makes use of results obtained in Section 3.

PROPOSITION 5.5. *If $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint confluent join CTRSs then $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$.*

Proof. ($\subseteq$) Trivial.

($\supseteq$) If $NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ is not a subset of $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ then there exists a smallest term t such that $t \in NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$ and $t \notin NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$. Clearly t must be a redex, so there is a rewrite rule $l \to r \Leftarrow s_1 \downarrow t_1, ..., s_n \downarrow t_n$ in $\mathscr{R}_1 \oplus \mathscr{R}_2$ and a substitution σ such that $t \equiv l^\sigma$ and $s_i^\sigma \downarrow t_i^\sigma$ for $i = 1, ..., n$. Note that $x^\sigma \in NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ for every $x \in \mathscr{D}(\sigma) \cap V(l)$, due to the minimality of t. Without loss of generality we assume that the rewrite rule stems from $\mathscr{R}_1$. We obtain $s_i^\sigma \downarrow_{1,2} t_i^\sigma$ for $i = 1, ..., n$ from Proposition 3.8 and Proposition 3.13 yields a substitution τ such that $\sigma \twoheadrightarrow_{1,2} \tau$ and $s_i^\tau \downarrow_1^\circ t_i^\tau$ $(i = 1, ..., n)$. Because $x^\tau \equiv x^\sigma$ for all $x \in V(l)$, we have $t \equiv l^\sigma \equiv l^\tau \to_1 r^\tau$, which contradicts the assumption $t \in NF(\mathscr{F}_1, \mathscr{R}_1)$. ∎

THEOREM 5.6. *Semi-completeness is a modular property of join CTRSs.*

Proof. Immediate consequence of Theorems 3.17 and 5.2 and Proposition 5.5. ∎

The non-left-linearity of $\mathscr{R}_1$ in Example 5.1 is not essential for the refutation of the modularity of weak normalization for join CTRSs. If we replace the first rule of $\mathscr{R}_1$ by

$$F(x, y) \to C \Leftarrow x \downarrow y,$$

we obtain a weakly normalizing join CTRS $\mathscr{R}'_1$ with the property that

$\mathscr{R}'_1 \oplus \mathscr{R}_2$ is not weakly normalizing, as is again witnessed by the term $F(b, c)$.

The following example shows that weak normalization is not a modular property of semi-equational CTRSs.

EXAMPLE 5.7. Let

$$\mathscr{R}_1 = \begin{cases} F(x, x) \rightarrow C \\ F(x, y) \rightarrow F(x, y) \Leftarrow x = y \end{cases}$$

and

$$\mathscr{R}_2 = \begin{cases} a \rightarrow b \\ a \rightarrow c. \end{cases}$$

Because $F(b, c)$ does not have a normal form, $\mathscr{R}_1 \oplus \mathscr{R}_2$ is not weakly normalizing, notwithstanding the weak normalization of both $\mathscr{R}_1$ and $\mathscr{R}_2$.

The proofs of the following results are very similar to the proofs of Theorem 5.2, Proposition 5.5, and Theorem 5.6.

THEOREM 5.8. *If* $(\mathscr{F}_1, \mathscr{R}_1)$ *and* $(\mathscr{F}_2, \mathscr{R}_2)$ *are disjoint weakly normalizing semi-equational CTRSs such that* $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$, *then* $\mathscr{R}_1 \oplus \mathscr{R}_2$ *is weakly normalizing.*

PROPOSITION 5.9. *If* $(\mathscr{F}_1, \mathscr{R}_1)$ *and* $(\mathscr{F}_2, \mathscr{R}_2)$ *are disjoint confluent semi-equational CTRSs then* $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{F}_\oplus, \mathscr{R}_1) \cap NF(\mathscr{F}_\oplus, \mathscr{R}_2)$.

THEOREM 5.10. *Semi-completeness is a modular property of semi-equational CTRSs.*

Example 5.7 shows that "no extra variables in the conditions" is not a sufficient condition for the modularity of weak normalization for semi-equational CTRSs. The modularity of weak normalization for left-linear semi-equational CTRSs cannot be refuted by adapting the first rule of $\mathscr{R}_1$ in Example 5.7. The next example, however, does the trick.

EXAMPLE 5.11. Let

$$\mathscr{R}_1 = \begin{cases} F(x) \rightarrow F(x) \Leftarrow x = C \\ F(C) \rightarrow D \end{cases}$$

and

$$\mathscr{R}_2 = \begin{cases} g(x) \rightarrow a \\ g(x) \rightarrow x. \end{cases}$$

Because $x =_{\mathscr{R}_1} C$ implies $x \equiv C$, $\mathscr{R}_1$ is weakly normalizing. The weak normalization of $\mathscr{R}_2$ is obvious. Both systems are left-linear, but the term $F(a)$ reduces only to itself since $a \leftarrow_{\mathscr{R}_2} g(C) \rightarrow_{\mathscr{R}_2} C$.

6. Unique Normal Forms

In Middeldorp (1989a) it is shown that UN is a modular property of TRSs. The proof is based on the fact that every TRS with unique normal forms can be conservatively extended to a confluent TRS with the same normal forms. This observation does not hold for join CTRSs, as is shown in the next example.

Example 6.1. Let

$$\mathscr{R} = \begin{cases} A \rightarrow B \\ A \rightarrow C \\ B \rightarrow B \\ C \rightarrow C \Leftarrow B \downarrow C. \end{cases}$$

Clearly $\mathscr{R}$ has the property UN. However, there does not exist a confluent join CTRS $\mathscr{R}'$ such that $\mathscr{R} \subseteq \mathscr{R}'$ and the normal forms of $\mathscr{R}$ and $\mathscr{R}'$ coincide. If such an $\mathscr{R}'$ were to exist then $B \downarrow_{\mathscr{R}'} C$ and therefore $C \rightarrow_{\mathscr{R}'} C$ which contradicts the equality of $NF(\mathscr{R})$ and $NF(\mathscr{R}')$.

It is an open problem whether the modularity of unique normal forms for join CTRSs can be obtained by some other method. In the remainder of this section we show that UN is a modular property of semi-equational CTRSs. First we show that every semi-equational CTRS with unique normal forms can be extended to a confluent semi-equational CTRS with the same normal forms. Our construction is a considerable simplification of the one in Middeldorp (1989a). For instance, we will see that it is sufficient to add at most one new constant whereas in Middeldorp (1989a) we employed infinitely many new function symbols. In Middeldorp (1990b) it is shown that this new construction enables a positive answer to a conjecture in Middeldorp (1989a) stating that the normal form property is a modular property of left-linear TRSs.

Let $(\mathscr{F}, \mathscr{R})$ be a semi-equational CTRS with unique normal forms. First we consider the case that $\mathscr{F}$ contains at least one constant symbol. We show that every equivalence class C of convertible terms contains a term t which can be used as a "common reduct" in order to obtain confluence with respect to C.

DEFINITION 6.2. (1) The set of equivalence classes of convertible terms is denoted by $\mathscr{C}$:

$$\mathscr{C} = \{\varnothing \neq C \subseteq \mathscr{T}(\mathscr{F}, \mathscr{V}) \mid C \text{ is closed under } =_{\mathscr{R}}\}.$$

(2) The subset of $\mathscr{C}$ consisting of all equivalence classes without a normal form is denoted by $\mathscr{C}^{\perp}$.

(3) If $C \in \mathscr{C}$ then $V_{\text{fix}}(C)$ denotes the set of variables occurring in every term $t \in C$:

$$V_{\text{fix}}(C) = \bigcap_{t \in C} V(t).$$

The next two propositions originate from Middeldorp (1989a). For the sake of completeness, the proofs are repeated here.

PROPOSITION 6.3. *If* $t \in C \in \mathscr{C}$ *and* $V(t) - V_{\text{fix}}(C) = \{x_1, ..., x_n\}$ *then* $t[x_i \leftarrow s_i \mid 1 \leqslant i \leqslant n] \in C$ *for all terms* $s_1, ..., s_n \in \mathscr{T}(\mathscr{F}, \mathscr{V})$.

Proof. We first prove the statement for all terms $s_1, ..., s_n \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ with $V(s_i) \cap \{x_1, ..., x_n\} = \varnothing$ for $i = 1, ..., n$. Define a sequence of terms $t_0, ..., t_n$ as follows:

$$t_i = \begin{cases} t & \text{if} \quad i = 0, \\ t_{i-1}[x_i \leftarrow s_i] & \text{if} \quad 1 \leqslant i \leqslant n. \end{cases}$$

We show that $t_i =_{\mathscr{R}} t$ by induction on i. The case $i = 0$ is trivial. Suppose the statement is true for all $i < k$ $(k > 0)$. Because $x_k \notin V_{\text{fix}}(C)$ there exists a term $u \in C$ such that $x_k \notin V(u)$. The induction hypothesis tells us that $t_{k-1} =_{\mathscr{R}} t$. This implies that

$$t_k \equiv t_{k-1}[x_k \leftarrow s_k] =_{\mathscr{R}} u[x_k \leftarrow s_k] \equiv u =_{\mathscr{R}} t.$$

Thus $t_n \equiv t[x_1 \leftarrow s_1] \cdots [x_n \leftarrow s_n] \equiv t[x_i \leftarrow s_i \mid 1 \leqslant i \leqslant n] \in C$. Now let $s_1, ..., s_n$ be arbitrary terms of $\mathscr{T}(\mathscr{F}, \mathscr{V})$. Choose distinct fresh variables $y_1, ..., y_n$. By the above argument we have $t[x_i \leftarrow y_i \mid 1 \leqslant i \leqslant n] \in C$ and because

$$V(t[x_i \leftarrow y_i \mid 1 \leqslant i \leqslant n]) - V_{\text{fix}}(C) = \{y_1, ..., y_n\}$$

we obtain $t[x_i \leftarrow y_i \mid 1 \leqslant i \leqslant n][y_i \leftarrow s_i \mid 1 \leqslant i \leqslant n] \equiv t[x_i \leftarrow s_i \mid 1 \leqslant i \leqslant n] \in C$. ∎

PROPOSITION 6.4. *If* $C \in \mathscr{C}$ *contains a normal form* t *then* $V_{\text{fix}}(C) = V(t)$.

Proof. Let $s \in C$. We show that $V(t) \subseteq V(s)$ by induction on the length of the conversion $s =_{\mathscr{R}} t$. The case of zero length is trivial. Let $s \leftrightarrow_{\mathscr{R}} s_1 =_{\mathscr{R}} t$. From the induction hypothesis we obtain $V(t) \subseteq V(s_1)$. If $s \rightarrow_{\mathscr{R}} s_1$ then $V(s_1) \subseteq V(s)$ and we are done. Assume $s \leftarrow_{\mathscr{R}} s_1$. We have to show that every variable of t occurs in s. Suppose to the contrary that there is a variable $x \in V(t)$ which does not occur in s. Choose a fresh variable y. Replacing every occurrence of x in the conversion $s_1 =_{\mathscr{R}} t$ by y yields a conversion $s_1' =_{\mathscr{R}} t'$. Note that t' is a normal form of $\mathscr{R}$ different from t. Because $x \notin V(s)$ we obtain $s_1' \rightarrow_{\mathscr{R}} s$. But now we have the conversion between t and t'

$$t =_{\mathscr{R}} s_1 \rightarrow_{\mathscr{R}} s \leftarrow_{\mathscr{R}} s_1' =_{\mathscr{R}} t',$$

which is impossible since $\mathscr{R}$ has unique normal forms. We conclude that $V_{\text{fix}}(C) = V(t)$. ∎

The following proposition is not true if $\mathscr{F}$ does not contain constant symbols.

PROPOSITION 6.5. *For every $C \in \mathscr{C}^{\perp}$ there exists a term $t \in C$ such that $V_{\text{fix}}(C) = V(t)$.*

Proof. Take an arbitrary term $s \in C$ and suppose that $V(s) - V_{\text{fix}}(C) = \{y_1, ..., y_m\}$. Let $t \equiv s[y_i \leftarrow c \mid 1 \leqslant i \leqslant m]$, where c is any ground term. Proposition 6.3 yields $t \in C$ and we have $V_{\text{fix}}(C) = V(t)$ by construction. ∎

According to the previous propositions we can define a mapping $\pi: \mathscr{C} \rightarrow \mathscr{T}(\mathscr{F}, \mathscr{V})$ with the following properties:

(1) $\pi(C) \in C$,

(2) if $C \in \mathscr{C}$ contains the normal form t then $\pi(C) \equiv t$,

(3) $V_{\text{fix}}(C) = V(\pi(C))$.

The term $\pi(C)$ serves as a common reduct for C.

DEFINITION 6.6. The TRS $(\mathscr{F}, \mathscr{R}')$ is defined by $\mathscr{R}' = \mathscr{R} \cup \{t \rightarrow \pi(C) \mid t \in C \in \mathscr{C} \text{ and } t \not\equiv \pi(C)\}$. Due to the above properties of π, $\mathscr{R}'$ contains only legal rewrite rules.

The reader is invited to check that the proof of parts (1) and (2) of the next proposition fails for join CTRSs.

PROPOSITION 6.7. (1) *For all terms $s, t \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ we have $s =_{\mathscr{R}} t$ if and only if $s =_{\mathscr{R}'} t$.*

(2) *$NF(\mathscr{F}, \mathscr{R}) = NF(\mathscr{F}, \mathscr{R}')$.*

(3) *The TRS $(\mathscr{F}, \mathscr{R}')$ is confluent.*

Proof. (1) If $s =_{\mathscr{R}} t$ then $s =_{\mathscr{R}'} t$ since $\mathscr{R}'$ is an extension of $\mathscr{R}$. For the other direction it is sufficient to prove that $s \to_{\mathscr{R}'} t$ implies $s =_{\mathscr{R}} t$. This will be done by induction on the depth of $s \to_{\mathscr{R}'} t$. If the depth equals zero then there exist an unconditional rewrite rule $l \to r \in \mathscr{R}'$, a context $C[\ ]$, and a substitution σ such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. If $l \to r \in \mathscr{R}$ then we clearly have $s \to_{\mathscr{R}} t$. Otherwise $r \equiv \pi(C)$ with $l \in C \in \mathscr{C}$ and we obtain $l =_{\mathscr{R}} r$ and hence $s =_{\mathscr{R}} t$. If the depth of $s \to_{\mathscr{R}'} t$ equals $n+1$ $(n \geqslant 0)$ then there exist a context $C[\ ]$, a conditional rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_m = t_m \in \mathscr{R}$, and a substitution σ such that $s \equiv C[l^\sigma]$, $t \equiv C[r^\sigma]$, and $s_i^\sigma =_{\mathscr{R}'} t_i^\sigma$ for $i = 1, \ldots, m$ with depth less than or equal to n. Note that $\mathscr{R}' - \mathscr{R}$ only contains unconditional rewrite rules. A straightforward induction on the length of the conversion $s_i^\sigma =_{\mathscr{R}'} t_i^\sigma$ yields $s_i^\sigma =_{\mathscr{R}} t_i^\sigma$ for $i = 1, \ldots, m$. Therefore $l^\sigma \to_{\mathscr{R}} r^\sigma$ and hence $s \to_{\mathscr{R}} t$.

(2) The inclusion $NF(\mathscr{F}, \mathscr{R}') \subseteq NF(\mathscr{F}, \mathscr{R})$ is evident. Suppose there exists a term $t \in \mathscr{T}(\mathscr{F}, \mathscr{V})$ such that $t \in NF(\mathscr{F}, \mathscr{R})$ and $t \notin NF(\mathscr{F}, \mathscr{R}')$. One easily shows that t cannot be reducible with respect to a rewrite rule of $\mathscr{R}' - \mathscr{R}$. Hence there exist a context $C[\ ]$, a substitution σ, and a rewrite rule $l \to r \Leftarrow s_1 = t_1, \ldots, s_n = t_n \in \mathscr{R}$ $(n \geqslant 0)$ such that $t \equiv C[l^\sigma]$ and $s_i^\sigma =_{\mathscr{R}'} t_i^\sigma$ for $i = 1, \ldots, n$. Part (1) shows that $s_i^\sigma =_{\mathscr{R}} t_i^\sigma$ for $i = 1, \ldots, n$ which implies $t \to_{\mathscr{R}} C[r^\sigma]$, contradicting the assumption $t \in NF(\mathscr{F}, \mathscr{R})$. We conclude that $NF(\mathscr{F}, \mathscr{R}) = NF(\mathscr{F}, \mathscr{R}')$.

(3) Suppose $s =_{\mathscr{R}'} t$. According to (1), s and t belong to the same class C of $\mathscr{R}$-convertible terms. By definition, both terms rewrite in zero or one step to their common reduct $\pi(C)$. ∎

We obtain the following result.

LEMMA 6.8. *Every semi-equational CTRS $(\mathscr{F}, \mathscr{R})$ with unique normal forms can be extended to a confluent CTRS $(\mathscr{F}', \mathscr{R}')$ such that:*

(1) *for all terms $s, t \in \mathscr{T}(\mathscr{F}', \mathscr{V})$ we have $s =_{\mathscr{R}} t$ if and only if $s =_{\mathscr{R}'} t$,*

(2) $NF(\mathscr{F}, \mathscr{R}) = NF(\mathscr{F}', \mathscr{R}')$.

Proof. If $\mathscr{F}$ contains a constant symbol then the preceding definitions and propositions yield the desired result. So assume that $\mathscr{F}$ only contains function symbols with arity greater than 0. Let $\bot$ be a fresh constant symbol and define $\mathscr{F}_1 = \mathscr{F} \cup \{\bot\}$ and $\mathscr{R}_1 = \mathscr{R} \cup \{\bot \to \bot\}$. The normal forms of $(\mathscr{F}, \mathscr{R})$ and $(\mathscr{F}_1, \mathscr{R}_1)$ clearly coincide. The equivalence of $=_{\mathscr{R}}$ and $=_{\mathscr{R}_1}$ with respect to $\mathscr{T}(\mathscr{F}_1, \mathscr{V})$ is also easily proved. Hence $(\mathscr{F}_1, \mathscr{R}_1)$ has unique normal forms. Because $\mathscr{F}_1$ contains a constant symbol, we know already the existence of a confluent semi-equational CTRS $(\mathscr{F}_1, \mathscr{R}_1')$ such that the relations $=_{\mathscr{R}_1}$ and $=_{\mathscr{R}_1'}$ coincide and $NF(\mathscr{F}_1, \mathscr{R}_1) = NF(\mathscr{F}, \mathscr{R}_1')$. Therefore

$s =_{\mathscr{R}} t$ if and only if $s =_{\mathscr{R}'_1} t$ for all terms $s, t \in \mathscr{T}(\mathscr{F}_1, \mathscr{V})$ and $NF(\mathscr{F}, \mathscr{R}) = NF(\mathscr{F}_1, \mathscr{R}'_1)$. ∎

The modularity of UN for TRSs is also based on the following result: if $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are disjoint TRSs and $(\mathscr{F}'_i, \mathscr{R}'_i)$ is an extension of $(\mathscr{F}_i, \mathscr{R}_i)$ with the same set of normal forms for $i = 1, 2$ such that $\mathscr{F}'_1 \cap \mathscr{F}'_2 = \varnothing$, then $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) = NF(\mathscr{R}'_1 \oplus \mathscr{R}'_2)$. The next example shows that this property is not true for semi-equational CTRSs.

EXAMPLE 6.9. Let $\mathscr{F}_1 = \mathscr{F}'_1 = \{a, b, c\}$, $\mathscr{F}_2 = \mathscr{F}'_2 = \{F, C\}$, $\mathscr{R}_1 = \{a \to b\}$, $\mathscr{R}'_1 = \mathscr{R}_1 \cup \{a \to c\}$ and $\mathscr{R}_2 = \mathscr{R}'_2 = \{F(x, y) \to C \Leftarrow x = y\}$. The term $F(b, c)$ belongs to $NF(\mathscr{R}_1 \oplus \mathscr{R}_2)$ because b and c are not convertible with respect to $\mathscr{R}_1 \oplus \mathscr{R}_2$. However, we have $F(b, c) \to_{\mathscr{R}'_1 \oplus \mathscr{R}'_2} C$ since $b \leftarrow_{\mathscr{R}_1} a \to_{\mathscr{R}'_1} c$. Therefore $NF(\mathscr{R}_1 \oplus \mathscr{R}_2) \neq NF(\mathscr{R}'_1 \oplus \mathscr{R}'_2)$ even though both $NF(\mathscr{F}_1, \mathscr{R}_1) = NF(\mathscr{F}'_1, \mathscr{R}'_1)$ and $NF(\mathscr{F}_2, \mathscr{R}_2) = NF(\mathscr{F}'_2, \mathscr{R}'_2)$. Note that $\mathscr{R}'_1$ is not confluent.

Fortunately, we will see that it is sufficient to prove the above-mentioned property only for confluent extensions.

PROPOSITION 6.10. *Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ be semi-equational CTRSs with the same set of normal forms. If $(\mathscr{F}_2, \mathscr{R}_2)$ is confluent and $\mathscr{F}'$ is a set of fresh function symbols then $NF(\mathscr{F}_1 \cup \mathscr{F}', \mathscr{R}_1) \subseteq NF(\mathscr{F}_2 \cup \mathscr{F}', \mathscr{R}_2)$.*

Proof. If $NF(\mathscr{F}_1 \cup \mathscr{F}', \mathscr{R}_1)$ is not a subset of $NF(\mathscr{F}_2 \cup \mathscr{F}', \mathscr{R}_2)$ then there exists a smallest term t such that $t \in NF(\mathscr{F}_1 \cup \mathscr{F}', \mathscr{R}_1)$ and $t \notin NF(\mathscr{F}_2 \cup \mathscr{F}', \mathscr{R}_2)$. First we show that $t \in \mathscr{T}((\mathscr{F}_1 \cap \mathscr{F}_2) \cup \mathscr{F}', \mathscr{V})$. Suppose to the contrary that $t \equiv C[F(t_1, ..., t_n)]$ for some n-ary function symbol $F \in \mathscr{F}_1 - \mathscr{F}_2$. Let $x_1, ..., x_n$ be distinct fresh variables. The term $F(x_1, ..., x_n)$ does not belong to $NF(\mathscr{F}_2, \mathscr{R}_2)$ and because $F(x_1, ..., x_n) \in \mathscr{T}(\mathscr{F}_1, \mathscr{V})$ and $NF(\mathscr{F}_1, \mathscr{R}_1) = NF(\mathscr{F}_2, \mathscr{R}_2)$, it must be $\mathscr{R}_1$-reducible. But then $C[F(t_1, ..., t_n)] \notin NF(\mathscr{F}_1 \cup \mathscr{F}', \mathscr{R}_1)$. Hence $t \in \mathscr{T}((\mathscr{F}_1 \cap \mathscr{F}_2) \cup \mathscr{F}', \mathscr{V})$. Combining this with the minimality of t and the assumption that $t \notin NF(\mathscr{F}_2 \cup \mathscr{F}', \mathscr{R}_2)$ yields a rewrite rule $l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n \in \mathscr{R}_2$ and a substitution σ such that $t \equiv l^\sigma$ and $s_i^\sigma =_{\mathscr{R}_2} t_i^\sigma$ for $i = 1, ..., n$. In the remainder of the proof we consider the disjoint union of the semi-equational CTRSs $(\mathscr{F}_2, \mathscr{R}_2)$ and $(\mathscr{F}', \varnothing)$. Because both systems are confluent we may use the results obtained in Section 3. We obtain $s_i^\sigma \downarrow_2 t_i^\sigma$ for $i = 1, ..., n$ from Proposition 3.8 (rephrased to the semi-equational case). Proposition 3.20 yields a substitution τ such that $\sigma \twoheadrightarrow_2 \tau$ and $s_i^\tau =_2^{\circ} t_i^\tau$ for $i = 1, ..., n$. Due to the minimality of t, $x^\sigma \in NF(\mathscr{F}_2 \cup \mathscr{F}', \mathscr{R}_2)$ for all $x \in V(l)$. Hence $x^\tau \equiv x^\sigma$ for all $x \in V(l)$ and thus $t \equiv l^\tau \to_2^{\circ} r^\tau$. Proposition 2.23 yields a decomposition $\tau_2 \circ \tau_1$ of τ such that τ_1 is black, τ_2 is top white, and $\tau_2 \propto \varepsilon$. (Remember that black corresponds to $\mathscr{F}_2$ and white to $\mathscr{F}'$.) Applying Proposition 3.19

gives us $\tau_1(l) \to_2^\circ \tau_1(r)$ and since $\tau_1(l)$, $\tau_1(r) \in \mathcal{T}(\mathcal{F}_2, \mathcal{V})$ we obtain $\tau_1(l) \notin NF(\mathcal{F}_2, \mathcal{R}_2)$, Hence $\tau_1(l) \notin NF(\mathcal{F}_1, \mathcal{R}_1)$ and thus $t \equiv \tau_2(\tau_1(l)) \notin NF(\mathcal{F}_1 \cup \mathcal{F}', \mathcal{R}_1)$, contradicting our assumption. We conclude that $NF(\mathcal{F}_1 \cup \mathcal{F}', \mathcal{R}_1) \subseteq NF(\mathcal{F}_2 \cup \mathcal{F}', \mathcal{R}_2)$. ∎

PROPOSITION 6.11. *Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint semi-equational CTRSs. If $(\mathcal{F}'_i, \mathcal{R}'_i)$ is a confluent extension of $(\mathcal{F}_i, \mathcal{R}_i)$ with the same set of normal forms for $i = 1, 2$ and $\mathcal{F}'_1 \cap \mathcal{F}'_2 = \varnothing$ then $NF(\mathcal{R}_1 \oplus \mathcal{R}_2) = NF(\mathcal{R}'_1 \oplus \mathcal{R}'_2)$.*

Proof. Let $\mathcal{F}'_\oplus = \mathcal{F}'_1 \cup \mathcal{F}'_2$. Because $\mathcal{R}_1 \cup \mathcal{R}_2$ is a subset of $\mathcal{R}'_1 \cup \mathcal{R}'_2$ we have $NF(\mathcal{R}'_1 \oplus \mathcal{R}'_2) \subseteq NF(\mathcal{F}'_\oplus, \mathcal{R}_1 \cup \mathcal{R}_2)$. It is not difficult to see that $NF(\mathcal{F}'_\oplus, \mathcal{R}_1 \cup \mathcal{R}_2) = NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$. For the other inclusion we assume that $t \in NF(\mathcal{R}_1 \oplus \mathcal{R}_2)$. Clearly $t \in NF(\mathcal{F}_\oplus, \mathcal{R}_1)$ and $t \in NF(\mathcal{F}_\oplus, \mathcal{R}_2)$. From Proposition 6.10 we obtain $t \in NF(\mathcal{F}'_1 \cup \mathcal{F}_2, \mathcal{R}'_1)$ and hence $t \in NF(\mathcal{F}'_\oplus, \mathcal{R}'_1)$. Likewise $t \in NF(\mathcal{F}'_\oplus, \mathcal{R}'_2)$. Therefore $t \in NF(\mathcal{R}'_1 \oplus \mathcal{R}'_2)$ by Proposition 5.9. ∎

Putting all pieces together, we obtain the modularity of unique normal forms for semi-equational CTRSs.

THEOREM 6.12. *UN is a modular property of semi-equational CTRSs.*

Proof. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be disjoint semi-equational CTRSs. We have to show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ has unique normal forms if and only if both $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ have unique normal forms.

$(\Rightarrow)$ Trivial.

$(\Leftarrow)$ According to Lemma 6.8 we may extend $(\mathcal{F}_i, \mathcal{R}_i)$ to a confluent CTRS $(\mathcal{F}'_i, \mathcal{R}'_i)$ with the same set of normal forms for $i = 1, 2$. Without loss of generality we assume that $\mathcal{F}'_1 \cap \mathcal{F}'_2 = \varnothing$. Let $s =_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ be a conversion between normal forms of $\mathcal{R}_1 \oplus \mathcal{R}_2$. Clearly $s =_{\mathcal{R}'_1 \oplus \mathcal{R}'_2} t$. According to Proposition 6.11 s and t are also normal forms with respect to $\mathcal{R}'_1 \oplus \mathcal{R}'_2$. Theorem 3.22 now yields the desired $s \equiv t$. ∎

7. CONCLUDING REMARKS

In this paper we studied the modular aspects of join and semi-equational CTRSs, but we did not pay attention to normal CTRSs. Since every normal CTRS can be viewed as a join CTRS, all positive results obtained for join CTRSs also hold for normal CTRSs. For instance, confluence and semi-completeness are modular properties of normal CTRSs. However, several counterexamples relating to join CTRSs involve a join CTRS which cannot be viewed as a normal CTRS. In particular, the modularity of local confluence and weak normalization for normal CTRSs should be investigated.

Another point which needs investigation is the syntactic restrictions imposed on the rewrite rules of CTRSs. From a programming point of view the assumption of a rewrite rule $l \to r \Leftarrow s_1 = t_1, ..., s_n = t_n$ satisfying the requirement that r only contains variables occurring in l is too restrictive. The CTRSs $\mathscr{R}$ we are interested in can be characterized by the phrase "if $s \to_{\mathscr{R}} t$ then $s \to t$ is a legal unconditional rewrite rule." However, the proofs in the preceding sections cannot easily be modified to cover these systems. For instance, Proposition 2.15 is no longer true.

In Middeldorp (1989b) it is shown that the union of two strongly normalizing TRSs is strongly normalizing if one of the TRSs contains neither collapsing nor duplicating rules (Theorem 2.6(3)) and in Example 4.8 we observed that join CTRSs do not satisfy this property. By imposing confluence on both systems we were able to retrieve the result for join CTRSs (Theorem 4.31). However, in Example 4.8 only the system with collapsing rules lacks confluence. Therefore we conjecture that the disjoint union of two strongly normalizing join CTRSs is strongly normalizing if one of them contains neither collapsing nor duplicating rules and the other is confluent.

The applicability of the results obtained in the previous chapters is rather limited due to the disjointness requirement. For combinations of TRSs which possibly share function symbols some results have been obtained, see Dershowitz (1981), Geser (1990), Kurihara and Ohuchi (1990), Middeldorp and Toyama (1991), and Toyama (1988). It is worthwhile to consider also combinations of CTRSs with shared function symbols.

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